Consistent Aggregation With Superlative and Other Price Indices

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1 Motivation and Background

- Often we want to decompose the overall inflation into sector specific inflation rates.
- For example, central banks decompose the overall inflation into the
  - core inflation (all products except energy and seasonal food) and the
  - non-core inflation (seasonal food and energy).
- A price index should give the same result with and without decomposition.
- Then the price index is said to be *consistent in aggregation*.

- A very restrictive notion of consistency in aggregation has been introduced by Vartia (1976a, b).
- Blackorby and Primont (1980) develop a far less restrictive version.
- Auer (2004) proposes a compromise between Vartia and Blackorby/Primont.
- Balk (1995, 1996) and Pursiainen (2005, 2008) advocate reverting to Vartia's restrictive version.

2 Basic Principle of Two Stage Aggregation

- Set of items: $S = (1, ..., N)$.
- Laspeyres index:

$$P^{\mathrm{La}} = \sum_{i \in S} r_i \frac{v_i^0}{\sum_{j \in S} v_j^0}$$

with $r_i = p_i^1 / p_i^0$ and $v_i^0 = p_i^0 q_i^0$.

- When $N = 1$, then all sensible price indices give $P = r_1$.
- Therefore, r_i is denoted as the primary attribute of the price index (Blackorby and Primont, 1980).
- The other attributes of a price index are secondary attributes (denoted by z_i^1, z_i^2, ...)
- The Laspeyres index has only one secondary attribute $z_i^1 = v_i^0$.

$(r_1,\dots,r_N),\ (v_1^0,\dots,v_N^0),\ (v_1^1,\dots,v_N^1)$	$P^{\mathrm{La}} = \sum_{i\in S} r_i \frac{v_i^0}{\sum_{j\in S} v_j^0}$
$(r_1,\dots,r_N),\ (z_1^1,\dots,z_N^1)$	$z_i^1 = v_i^0$
single stage compilation	$P^{\mathrm{La}} = \sum_{i\in S} r_i \frac{z_i^1}{\sum_{j\in S} z_j^1}$
two stage compilation $S = (S_1,\dots,S_K)$	
for $k = 1,\dots,K$:	$P_k^{\mathrm{La}} = \sum_{i\in S_k} r_i \frac{z_i^1}{\sum_{j\in S_k} z_j^1}$
for $k = 1,\dots,K$:	$Z_k^1 = \sum_{i\in S_k} z_i^1$
$(P_1^{\mathrm{La}},\dots,P_K^{\mathrm{La}}),\ (Z_1^1,\dots,Z_K^1)$	
	$P^{\mathrm{La}} = \sum_{k=1}^{K} P_k^{\mathrm{La}} \frac{Z_k^1}{\sum_{l=1}^{K} Z_l^1}$

3 Illustrative Example

- Swedish CPI Data from the base period 2010 $(t = 0)$ and the comparison period 2011 $(t = 1)$.
- $S = 1, 2, \ldots, 360$ items (four-digit level COICOP classification)
- $S_1 = 1, 2, \ldots, 301$ are the items assigned to core inflation.
- $S_2 = 302, \ldots, 360$ are the items assigned to non-core inflation.
- For each item we know (r_i, v_i^0, v_i^1).

Table 1: Two Stage Aggregation of Laspeyres Index

	Basic Heading Information							Sec. Attrib.
	i	Coicop	Group	Product	r_i	v_i^0	v_i^1	$z_i^1 = v_i^0$
S_1 (Core)	1	01.1.1	1113	Wheat Bread	1.0333	1524	1562	1524
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	301	12.7	9704	Lawyer Fees	1.0282	1067	1085	1067
					$P_1^{\text{La}} = 1.0268$			$Z_1^1 = 1\,243\,742$
S_2 (Other)	302	01.1.3	1307	Herring	1.0438	155	128	155
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	360	07.2.2	6225	E 85 Fuel	1.0479	1205	1245	1205
					$P_2^{\text{La}} = 1.0366$			$Z_2^1 = 182\,775$

∗ Source: Statistics Sweden, Consumer Price Index Data for 2010-2011.

- Single stage aggregation by the Laspeyres index:

$$P^{\mathrm{La}} = \sum_{i \in S} r_i \frac{z_i^1}{\sum_{j \in S} z_j^1} = 1.028025$$

with $z_i^1 = v_i^0$.

- Second stage of two stage aggregation by the Laspeyres index:

$$P^{\mathrm{La}} = \sum_{k=1,2} P_k^{\mathrm{La}} \frac{Z_k^1}{\sum_{l=1,2} Z_l^1} = 1.028025$$

- Laspeyres index is consistent in aggregation with respect to the secondary attribute $z_i^1 = v_i^0$.

4 Superlative Price Indices

- Fisher index

$$P^{\mathrm{Fi}} = \left(\frac{\sum_{i \in S} v_i^0 r_i}{\sum_{i \in S} v_i^0} \frac{\sum_{i \in S} v_i^1}{\sum_{i \in S} v_i^1 / r_i} \right)^{1/2}$$

- Törnqvist index

$$\ln P^{\mathrm{T\ddot{o}}} = \sum_{i \in S} \ln(r_i) \frac{1}{2} \left(\frac{v_i^0}{\sum_{j \in S} v_j^0} + \frac{v_i^1}{\sum_{j \in S} v_j^1} \right)$$

- Walsh index

$$P^{\mathrm{Wa}} = \sum_{i \in S} r_i \frac{\sqrt{v_i^0 v_i^1 / r_i}}{\sum_{j \in S} \sqrt{v_j^0 v_j^1 / r_j}}$$

	$P^{\mathrm{Fi}} = \left(\frac{\sum_{i\in S} v_i^0 r_i}{\sum_{i\in S} v_i^0} \frac{\sum_{i\in S} v_i^1}{\sum_{i\in S} v_i^1 / r_i} \right)^{1/2}$ $z_i^1 = v_i^0, z_i^2 = v_i^1, z_i^3 = v_i^0 r_i, z_i^4 = v_i^1 / r_i$
single stage	$P^{\mathrm{Fi}} = \left(\frac{\sum_{i\in S} z_i^3}{\sum_{i\in S} z_i^1} \frac{\sum_{i\in S} z_i^2}{\sum_{i\in S} z_i^4} \right)^{1/2}$
two stage	$P_k^{\mathrm{Fi}} = \left(\frac{\sum_{i\in S_k} z_i^3}{\sum_{i\in S_k} z_i^1} \frac{\sum_{i\in S_k} z_i^2}{\sum_{i\in S_k} z_i^4} \right)^{1/2}$ $Z_k^1 = \sum_{i\in S_k} z_i^1, \ldots, Z_k^4 = \sum_{i\in S_k} z_i^4$ $P^{\mathrm{Fi}} = \left(\frac{\sum_{k=1}^K Z_k^3}{\sum_{k=1}^K Z_k^1} \frac{\sum_{k=1}^K Z_k^2}{\sum_{k=1}^K Z_k^4} \right)^{1/2}$

- Fisher index is consistent in aggregation with respect to $z_i^1 = v_i^0$, $z_i^2 = v_i^1$, $z_i^3 = v_i^0 r_i$, and $z_i^4 = v_i^1 / r_i$.
- Possible objections:
  - primary attribute is missing in index formula
  - $z_i^3 = z_i^1 r_i$, and $z_i^4 = z_i^2 / r_i$, but $Z_k^3 \neq Z_k^1 P_k$, and $Z_k^4 \neq Z_k^2 / P_k$.
  - secondary attributes must be either v_i^0 or v_i^1.

	$P^{\mathrm{Wa}} = \sum_{i \in S} r_i \frac{\sqrt{v_i^0 v_i^1 / r_i}}{\sum_{j \in S} \sqrt{v_j^0 v_j^1 / r_j}}$ $z_i^1 = \sqrt{v_i^0 v_i^1 / r}$
single stage	$P^{\mathrm{Wa}} = \sum_{i \in S} r_i \frac{z_i^1}{\sum_{j \in S} z_j^1}$
two stage	$P_k^{\mathrm{Wa}} = \sum_{i \in S_k} r_i \frac{z_i}{\sum_{j \in S_k} z_j}$ $Z_k^1 = \sum_{i \in S_k} z_i^1$ $P^{\mathrm{Wa}} = \sum_{k=1}^{K} P_k^{\mathrm{Wa}} \frac{Z_k}{\sum_{l=1}^{K} Z_l}$

- Walsh index is consistent in aggregation with respect to $z_i^1 = \sqrt{v_i^0 v_i^1 / r_i}$.
- Possible objections:
    - secondary attributes must be either v_i^0 or v_i^1.

5 Other Price Indices

Table 2: More Price Indices That Are Consistent in Aggregation

Name	Price Index Formula	Function $f(r_i, z_i^1, \ldots, z_i^Q)$	Secondary Attributes z_i^q
Laspeyres	$P^{\mathrm{La}} = \frac{\sum v_i^0 r_i}{\sum v_i^0}$	$r_i\, z_i$	v_i^0
Paasche	$P^{\mathrm{Pa}} = \frac{\sum v_i^1}{\sum v_i^1 / r_i}$	$r_i^{-1}\, z_i$	v_i^1
Marshall-Edgeworth	$P^{\mathrm{ME}} = \sum r_i \frac{v_i^0 + v_i^1/r_i}{\sum (v_i^0 + v_i^1/r_i)}$	$r_i\, z_i$	$(v_i^0 + v_i^1/r_i)$
Walsh-2	$\ln P^{\mathrm{Wa2}} = \sum \ln r_i \frac{\sqrt{v_i^0 v_i^1}}{\sum \sqrt{v_j^0 v_j^1}}$	$(\ln r_i)\, z_i$	$\sqrt{v_i^0 v_i^1}$

Table 2: (contin.)

Walsh-Vartia	$\ln P^{\mathrm{WV}} = \sum \ln r_i \frac{\sqrt{v_i^0}}{\sqrt{\sum v_j^0}} \frac{\sqrt{v_i^1}}{\sqrt{\sum v_j^1}}$	$(\ln r_i)\sqrt{z_i^1 z_i^2}$	$v_i^0,\ v_i^1$
Theil	$\ln P^{\mathrm{Th}} = \sum \ln r_i \frac{\sqrt[3]{\frac{1}{2}(v_i^0 + v_i^1) v_i^0 v_i^1}}{\sum \sqrt[3]{\frac{1}{2}(v_j^0 + v_j^1) v_j^0 v_j^1}}$	$(\ln r_i)\, z_i$	$\sqrt[3]{\frac{1}{2}(v_i^0 + v_i^1) v_i^0 v_i^1}$
Vartia	$\ln P^{\mathrm{Va}} = \sum \ln r_i \frac{L(v_i^0, v_i^1)}{L(\sum v_i^0, \sum v_i^1)}$ with* $L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a} & \text{for } a \neq b \\ a & \text{for } a = b \end{cases}$	$(\ln r_i)\, L(z_i^1, z_i^2)$	$v_i^0,\ v_i^1$

Table 3: Generalized Unit Value (GUV) Indices

Name	Price Index Formula	Function $f(r_i, z_i^1, \ldots, z_i^Q)$	Secondary Attributes z_i^q
Banerjee (GUV-3)**	$P^{\mathrm{Ba}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 (1 + r_i)}{\sum v_i^1 (1 + 1/r_i)}$	$r_i \, \frac{z_i^1}{z_i^2} \, z_i^3$	$v_i^0, v_i^1, v_i^1 \frac{1 + r_i}{r_i}$
Davies (GUV-4)**	$P^{\mathrm{Da}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 \sqrt{r_i}}{\sum v_i^1 \sqrt{1/r_i}}$	$r_i \, \frac{z_i^1}{z_i^2} \, z_i^3$	$v_i^0, v_i^1, v_i^1/\sqrt{r_i}$
(GUV-5)**	$P^{\mathrm{GUV\text{-}5}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 \left(1 + r_i^{-1}\right)^{-1}}{\sum v_i^1 \left(1 + r_i\right)^{-1}}$	$r_i \, \frac{z_i^1}{z_i^2} \, z_i^3$	$v_i^0, v_i^1, v_i^1/(r_i + 1)$

Table 3: (contin.)

(GUV-6)**	$P^{\text{GUV-6}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 r_i^{v_i^1/(v_i^0+v_i^1)}}{\sum v_i^1 r_i^{-v_i^0/(v_i^0+v_i^1)}}$	r_i $\frac{z_i^1}{z_i^2}$ z_i^3	$v_i^0, v_i^1, v_i^1 r_i^{\frac{-v_i^0}{v_i^0+v_i^1}}$
Lehr (GUV-7)**	$P^{\text{Le}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 (v_i^0 + v_i^1)(v_i^0 + v_i^1/r)}{\sum v_i^1 (v_i^0 + v_i^1)(v_i^0 r_i + v_i^1)}$	r_i $\frac{z_i^1}{z_i^2}$ z_i^3	$v_i^0, v_i^1, v_i^1 \frac{v_i^0 + v_i^1}{r_i v_i^0 + v_i^1}$

6 Additional Requirements

- In contrast to Blackorby and Primont (1980), we allow only for secondary attributes that are functions of no other information than r_i, v_i^0 and v_i^1.
- **Requirement A:** Secondary attributes should represent monetary values (e.g., v_i^0 or $\sqrt{v_i^0 v_i^1}$, but not $v_i^0 v_i^1$).
- **Requirement B:** The secondary attributes are aggregated additively: $Z_k^q = \sum_{i \in S_k} z_i^q$.
- **Requirement C:** Any functional relationship between the secondary attributes of the individual items must carry over to the aggregated secondary attributes.
- This eliminates the indices of Table 3, the Fisher index, but not the Walsh index.
- The Walsh index is "ABC-consistent in aggregation".

- **Requirement D:** (Auer, 2004) Only the secondary attributes v_i^0, v_i^1, $v_i^0 r_i$ and v_i^1/r_i are admissable (note that $v_i^0 r_i = p_i^1 q_i^0$ and $v_i^1/r_i = p_i^0 q_i^1$).
- This eliminates the Walsh, the Walsh-2, and the Theil index.
- **Requirement E:** (Vartia, 1976a,b, Balk 1995, Pursiainen 2005, 2008) Only the secondary attributes v_i^0 and v_i^1 are admissable.
- This eliminates the Marshall-Edgeworth index.
- Then we are left with the Laspeyres, Paasche, Walsh-Vartia, and Vartia index.

7 Concluding Remarks

- Very heterogeneous definitions of consistency in aggregation have been proposed in the literature.
- We have introduced a rigorous formalization of this notion that allows to compare these definitions.
- Our definition of consistency in aggregation can be made more restrictive by attaching additional requirements.
- The Walsh index satisfies the three least controversial of these requirements.