Dynamic Semiparametric Models for Expected Shortfall (and Value-at-Risk)

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Measures of market risk

- The simplest and most widely-used measure of risk is **variance**:

$$\sigma_t^2 \equiv \mathbb{E}_{t-1}\left[\left(Y_t - \mu_t\right)^2\right]$$

- In the 1990s, in part prompted by Basel I and II, attention in risk management moved to **Value-at-Risk:**

$$VaR_t \equiv F_t^{-1}(\alpha) \Rightarrow \Pr_{t-1}\left[Y_t \leq VaR_t\right] = \alpha$$

- The Basel III accord pushes banks to move from Value-at-Risk towards **Expected Shortfall:**

$$ES_t \equiv \mathbb{E}_{t-1}\left[Y_t | Y_t \leq VaR_t\right]$$

Why the move from VaR to ES?

- Academic work has highlighted some problems with VaR (see McNeil, *et al.* 2015 for a summary):
- Value-at-Risk has some positive attributes:
    - Focuses on the **left tail** of returns, so more relevant for risk mgmt
    - **Easy** to interpret ("the loss that is only exceeded on 5% of days")
    - Is well-defined even for **fat-tailed** distributions; is a **robust** statistic
- But VaR suffers from important drawbacks (Artzner et al. 1999, *MathFin*):
    - Not "sub-additive:" diversification may make VaR look **worse**
    - **No information** about losses beyond the VaR
- Expected Shortfall addresses both of these drawbacks
    - But it is not a robust statistic, and does require moment assumptions

Why aren't there more models for Expected Shortfall?

- To answer this, consider how we estimate and model Value-at-Risk.
- For a given sample $\{Y_t\}_{t=1}^{T}$, VaR can be obtained as

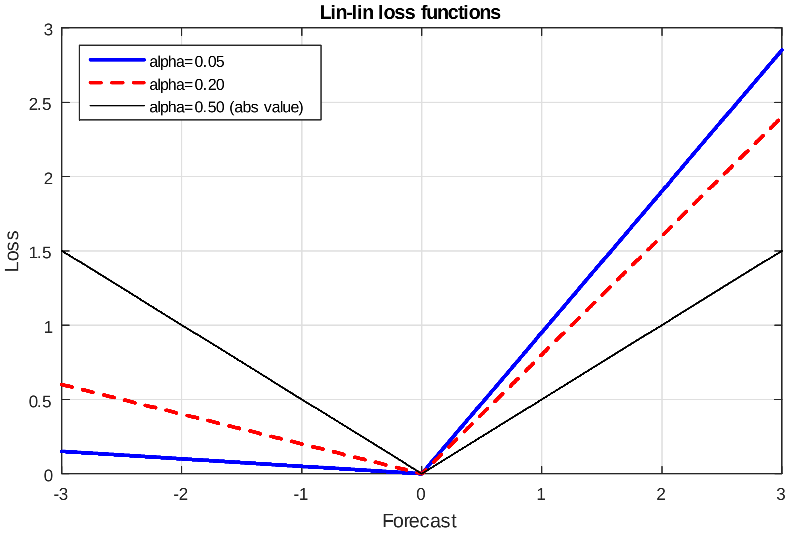
$$\widehat{VaR}_T = \arg\min_{v} \frac{1}{T} \sum_{t=1}^{T} L(Y_t, v; \alpha)$$

$$\text{where} \quad L(y, v; \alpha) = (\mathbf{1}\{y \leq v\} - \alpha)(v - y)$$

- The loss function here is the "tick" or "lin-lin" loss function
- Given this loss function, it is possible to consider models like "CAViaR" (Engle and Manganelli, 2004, *JBES*):

$$\begin{aligned} \hat{\boldsymbol{\theta}}_T &= \arg\min_{\boldsymbol{\theta}} \frac{1}{T} \sum_{t=1}^{T} L(Y_t, v(\mathbf{Z}_{t-1}; \boldsymbol{\theta}); \alpha) \\ \text{and} \quad VaR_t &= v(\mathbf{Z}_{t-1}; \boldsymbol{\theta}) \end{aligned}$$

The "lin-lin" loss function



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- Given an estimator of VaR, sample Expected Shortfall can be computed as:

$$\widehat{ES}_T = \frac{1}{\alpha T} \sum_{t=1}^{T} Y_t \mathbf{1}\{Y_t \le VaR_t\}$$

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- But there *does not exist* an objective function such that ES is the solution:

$$\nexists\ L^* \quad \text{s.t.} \quad \widehat{ES}_T = \arg\min_{e} \ \frac{1}{T} \sum\nolimits_{t=1}^{T} L^*\left(Y_t, e; \alpha \right)$$

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- Expected Shortfall is "**non-elicitable**" (Gneiting 2011, *JASA*). This explains, perhaps, the lack of models for Expected Shortfall:

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- Expected Shortfall is "**non-elicitable**" (Gneiting 2011, *JASA*). This explains, perhaps, the lack of models for Expected Shortfall:

★ We exploit recent results in statistics and decision theory which shows that while ES is not elicitable, it is **jointly elicitable** with Value-at-Risk.

Related literature

- A lot of work has been done on models for risk management, mostly VaR:
  - McNeil, Frey and Embrechts (2015, *Quantitative Risk Mgmt*)
  - Daníelsson (2011, *Financial Risk Forecasting*)
  - Komunjer (2010, *Handbook of Economic Forecasting*)
- This paper is closest to Engle and Manganelli (2004, *JBES*) who propose time series models for conditional quantiles, and establish conditions for estimation and inference
  - We extend their paper to consider ES (jointly with VaR)
- We draw on two distinct recent advances in the literature:
  - Statistical decision theory: Fissler and Ziegel (2016, AoS)
  - Parameter-driven time series models: Creal, Koopman and Lucas (2013, *JAE*)

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2. Estimating Expected Shortfall (and Value-at-Risk)
   - The Fissler-Ziegel loss function
   - Dynamic models for VaR and ES

3. Inference methods
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Joint estimation of VaR and Expected Shortfall

- Fissler and Ziegel (2016, AoS) show that while ES is not elicitable, it is **jointly elicitable** with VaR, using the class of "FZ" loss functions.

- We will use a homogeneous of degree zero FZ loss function, as for the values of α of interest we know $ES_t < 0$. There is only one such FZ loss:

$$L_{FZ0}(Y, v, e; \alpha) = -\frac{1}{\alpha e}\mathbf{1}\{Y \leq v\}(v - Y) - \frac{1}{e}(e - v) + \log(-e)$$

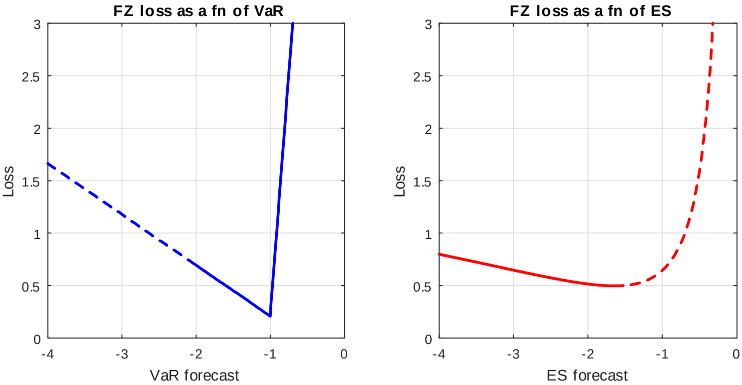
  - where Y is the (future) return, v is the VaR forecast, and e is the ES forecast.
  - This loss function yields loss function *differences* (between two competing sets of VaR and ES forecasts) thare homogeneous of degree zero.

- Minimizing this loss function yields VaR and ES:

$$[VaR_t, ES_t] = \arg\min_{(v,e)} \mathbb{E}_{t-1}[L_{FZ0}(Y_t, v, e; \alpha)]$$

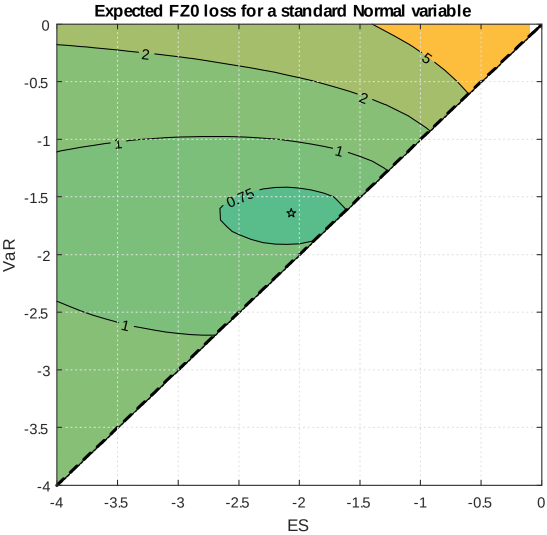
The FZ0 loss function

The implied VaR loss is the familiar "tick" loss function; the implied ES loss resembles "QLIKE"



The expected FZ0 loss function

for a N(0,1) target variable. Contours are convex.



Dynamic models for ES and VaR

- With a loss function available, it is possible to consider dynamic models for ES and VaR:

$$
\begin{aligned}
VaR_t &= v\left(\mathbf{Z}_{t-1};\boldsymbol{\theta}\right) \\
ES_t &= e\left(\mathbf{Z}_{t-1};\boldsymbol{\theta}\right)
\end{aligned}
$$

- The parameters of this model can then be obtained as:

$$
\hat{\boldsymbol{\theta}}_T = \arg\min_{\boldsymbol{\theta}} \ \frac{1}{T}\sum\nolimits_{t=1}^{T} L\left(Y_t, v\left(\mathbf{Z}_{t-1};\boldsymbol{\theta}\right), e\left(\mathbf{Z}_{t-1};\boldsymbol{\theta}\right)\right)
$$

- We propose some new models for ES (and VaR), drawing on recent research, and then provide theory for estimation and inference for these models.

GAS models for dynamic ES and VaR I

- Creal et al. (2013, *JAE*) proposed "generalized autoregressive score" models for time-varying density models:

$$\begin{aligned} Y_t|\mathcal{F}_{t-1} &\sim F(\boldsymbol{\theta}_t) \\ \boldsymbol{\theta}_t &= \mathbf{w} + \mathbf{B}\cdot\boldsymbol{\theta}_{t-1} + \mathbf{A}\cdot\mathbf{S}_{t-1}\frac{\partial \log f(y_{t-1};\boldsymbol{\theta}_{t-1})}{\partial\boldsymbol{\theta}} \end{aligned}$$

- Using the score ($\partial \log f/\partial\boldsymbol{\theta}$) as the "forcing variable" enables them to nest many existing models, including ARMA and GARCH models.
  - The "scale" matrix, $\mathbf{S}_{t-1}$, is often set to the inverse Hessian.
- This choice of forcing variable can be motivated as the Newton-Raphson step in a numerical optimization algorithm.

GAS models for dynamic ES and VaR II

- We adopt this modeling approach, and apply it to our M-estimation problem.
- Consider the following GAS(1,1) specification for VaR and ES:

$$\begin{aligned}\begin{bmatrix} v_{t+1} \\ e_{t+1} \end{bmatrix} &= \mathbf{w} + \mathbf{B} \begin{bmatrix} v_t \\ e_t \end{bmatrix} + \mathbf{A} \left(\frac{\partial^2 \mathbb{E}_{t-1} \left[L \left(Y_t, v_t, e_t \right) \right]}{\partial \left(\mathbf{ve} \right) \partial \left(\mathbf{ve} \right)'} \right)^{-1} \frac{\partial L \left(Y_t, v_t, e_t \right)}{\partial \left(\mathbf{ve} \right)} \\ &= \mathbf{w} + \mathbf{B} \begin{bmatrix} v_{t-1} \\ e_{t-1} \end{bmatrix} + \mathbf{A} \begin{bmatrix} \lambda_{v,t-1} \\ \lambda_{e,t-1} \end{bmatrix}\end{aligned}$$

where the "forcing variables" are given by

$$\begin{aligned}\lambda_{v,t} &= -v_t \left(\mathbf{1} \left\{ Y_t \leq v_t \right\} - \alpha \right) \\ \lambda_{e,t} &= - \left(\frac{1}{\alpha} \mathbf{1} \left\{ Y_t \leq v_t \right\} Y_t - e_t \right)\end{aligned}$$

Competing models I

- While there are relatively few dynamic models for ES, there are some. We consider the following models as competition:

1. **Rolling window**:

$$
\begin{aligned}
\widehat{VaR}_t &= \widehat{Quantile}\left\{Y_s\right\}_{s=t-m+1}^{t} \\
\widehat{ES}_t &= \frac{1}{\alpha m}\sum_{s=t-m+1}^{t} Y_s \mathbf{1}\left\{Y_s \leq \widehat{VaR}_s\right\}
\end{aligned}
$$

   - $m \in \{125, 250, 500\}$

Competing models II

2 ARMA-GARCH models

$$
\begin{aligned}
Y_t &= \mu_t + \sigma_t \eta_t \\
\mu_t &\sim ARMA(p,q) \quad , \quad \sigma_t^2 \sim GARCH(p,q)
\end{aligned}
$$

$$
\begin{array}{llll}
\text{a.} & \eta_t & \sim & iid\ N(0,1) \\
\text{b.} & \eta_t & \sim & iid\ Skew\ t(0,1,\nu,\lambda) \\
\text{c.} & \eta_t & \sim & iid\ F(0,1) \quad \text{(estimated by the EDF)}
\end{array}
$$

- Model 2(c) is also known as "filtered historical simulation," and is probably the best existing model for ES (see survey by Engle and Manganelli (2004, *book*)).

Pros and cons of directly modeling ES and VaR

- Consider a generic model:

$$
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VaR_t &= v\left(\mathbf{Z}_{t-1}; \boldsymbol{\theta}\right) \\
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- Such a model is a *semiparametric* model for returns:
    - We assume **parametric dynamics** for ES and VaR
    - We make **no assumptions about the distribution** of returns (beyond regularity conditions required for estimation and inference)

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- By eliminating the need for assumptions about the distribution of returns, we hopefully obtain a more robust model. But:
  - There may be **efficiency losses**. We will study this carefully in our OOS forecasting analysis.
  - This is not a **complete probability model**: further assumptions are needed to draw simulations, for example.

A one-factor model

- Consider a case where there is only one latent factor driving VaR and ES:

$$\begin{aligned} v_t &= a \exp\{\kappa_t\} \\ e_t &= b \exp\{\kappa_t\}, \quad \text{where } b < a < 0 \\ \text{where} \quad \kappa_t &= \omega + \beta\kappa_{t-1} + \gamma H_{t-1}^{-1} s_{t-1} \end{aligned}$$

- If we derive the GAS dynamics for κ_t we find

$$H_{t-1}^{-1} s_{t-1} = \frac{-1}{e_{t-1}} \left(\frac{1}{\alpha} \mathbf{1}\{Y_{t-1} \leq v_{t-1}\} Y_{t-1} - e_{t-1} \right) \equiv \frac{-\lambda_{e,t-1}}{e_{t-1}}$$

  - The intercept, ω, is not identified here so we fix it at zero.

GARCH with FZ estimation

- Next consider GARCH dynamics for the latent factor, but estimate using the FZ0 loss function rather than QML:

$$\begin{aligned} Y_t &= \kappa_t \eta_t, \quad \eta_t \sim iid\ F_\eta \\ \text{so} \quad v_t &= a \cdot \kappa_t \\ e_t &= b \cdot \kappa_t, \ \text{with } b < a < 0 \\ \text{and} \quad \kappa_t^2 &= \omega + \beta \kappa_{t-1}^2 + \gamma Y_{t-1}^2 \end{aligned}$$

  - As above, the intercept, ω, is not identified here and we fix it at one.

- If the GARCH model is **correct**, this is consistent but almost certainly less efficient than QML

- If the model is **misspecified**, estimating this way yields the parameters that lead to the best possible VaR and ES forecasts**.**

A hybrid GAS+GARCH model

- Finally, consider a "hybrid" model, where as before we have:

$$\begin{aligned} Y_t &= \exp\{\kappa_t\}\eta_t, \quad \eta_t \sim iid\ F_\eta \\ \text{so} \quad v_t &= a\exp\{\kappa_t\} \\ e_t &= b\exp\{\kappa_t\}, \quad \text{with } b < a < 0 \end{aligned}$$

- We augment the GAS dynamics for κ_t with a "GARCH" term:

$$\kappa_t = \omega + \beta\kappa_{t-1} + \gamma\frac{-\lambda_{e,t-1}}{e_{t-1}} + \delta\log|Y_{t-1}|$$

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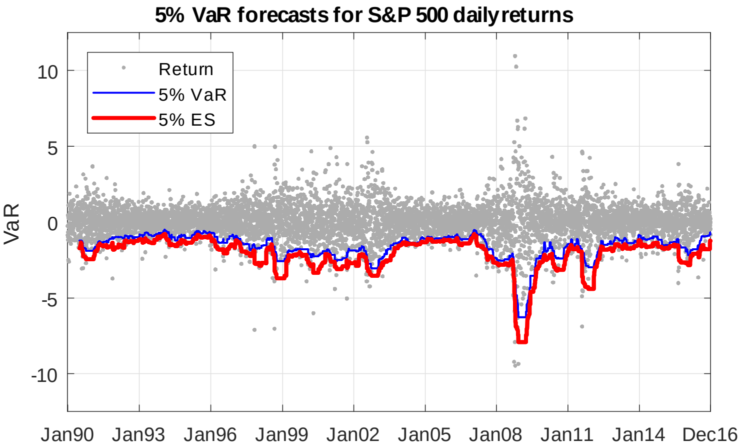
Data

- We study daily returns on four equity indices
  - S&P 500
  - Dow Jones Industrial Average
  - NIKKEI 225
  - FTSE 100.
- Sample period is January 1990 to December 2016
  - Number of observations (T) is 6630 to 6805.
  - We use the first 10 years ($R \approx 2500$) for estimation, and the last 17 years ($P \approx 4250$) for out-of-sample forecast comparison.

Daily returns on the S&P 500 index

Rolling window estimates of the 5% VaR and ES



Summary statistics

	S&P 500	DJIA	NIKKEI	FTSE
Mean (Annualized)	6.776	7.238	-2.682	3.987
Std dev (Annualized)	17.879	17.042	24.667	17.730
Skewness	-0.244	-0.163	-0.114	-0.126
Kurtosis	11.673	11.116	8.580	8.912
VaR-0.01	-3.128	-3.034	-4.110	-3.098
VaR-0.025	-2.324	-2.188	-3.151	-2.346
VaR-0.05	-1.731	-1.640	-2.451	-1.709
ES-0.01	-4.528	-4.280	-5.783	-4.230
ES-0.025	-3.405	-3.215	-4.449	-3.295
ES-0.05	-2.697	-2.553	-3.603	-2.643

ARMA-GARCH-Skew t models for these returns

	S&P 500	**DJIA**	**NIKKEI**	**FTSE**
Mean	ARMA(1,1)	AR(2)	AR(0)	AR(4)
R^2	0.006	0.004	0.000	0.009
ω	0.014	0.017	0.066	0.016
β	0.905	0.897	0.863	0.893
α	0.082	0.088	0.113	0.094
ν	6.934	7.062	7.806	11.800
λ	-0.115	-0.100	-0.066	-0.102

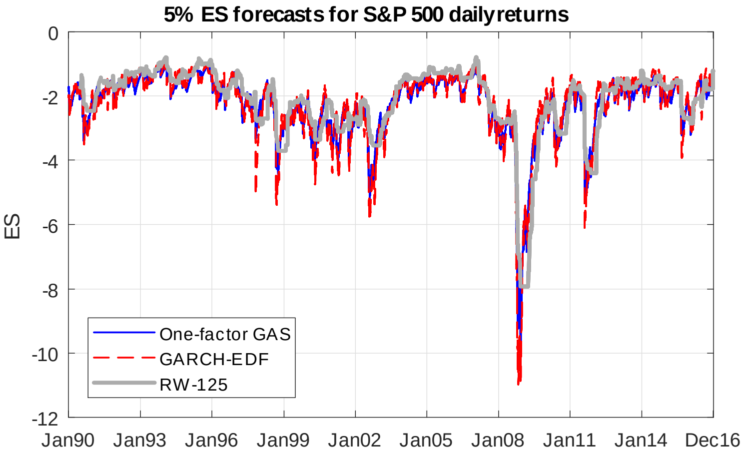
One-factor models for ES and VaR

SP500, alpha=0.05. Preferred model is the “hybrid” model

	GAS-1F	GARCH-FZ	Hybrid
β	0.990 (0.004)	0.908 (0.072)	0.968 (0.015)
γ	-0.010 (0.002)	0.030 (0.010)	-0.011 (0.002)
δ	–	–	0.018 (0.009)
a	-1.490 (0.346)	-2.659 (0.492)	-2.443 (0.473)
b	-2.089 (0.487)	-3.761 (0.747)	-3.389 (0.664)
Avg Loss	0.750	0.762	0.745

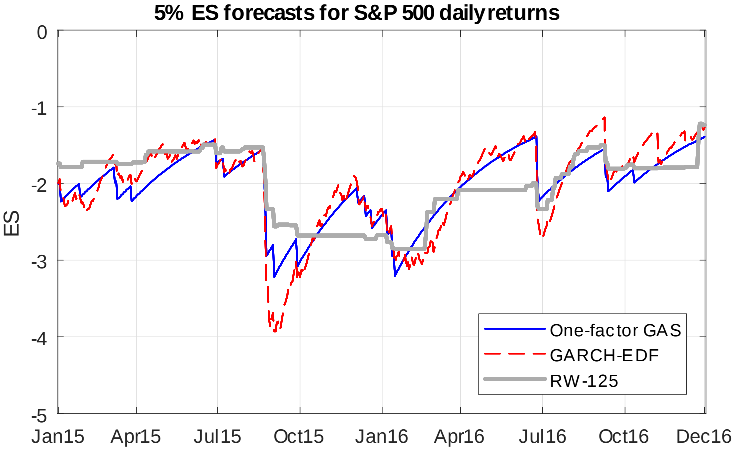
Dynamic Expected Shortfall: 1990-2016

ES ranges from around -1.5% in mid 90s, to -10% in financial crisis



Dynamic Expected Shortfall: 2015-2016

The difference between the GAS and GARCH forcing variables is apparent here



The models used for in OOS forecast comparison

- Rolling Window, with $m \in \{125, 250, 500\}$
- GARCH(1,1) with Normal, Skew t, or EDF for the residuals

- ★ GAS(1,1) dynamics, 2 factors
- ★ GAS(1,1) dynamics, 1 factor
- ★ GARCH-FZ: estimating the GARCH model using the FZ loss function
- ★ Hybrid model: one-factor GAS model, with GARCH forcing variable included

Evaluating and comparing out-of-sample forecasts

- We estimate the models using data only from the estimation sample (up until Dec 1999)
  - $R \approx 2500$, $P \approx 4250$
- Forecasts of VaR and ES are then produced for each day in the OOS period
  - No look-ahead bias in the forecasts
- We compare the forecasts using the FZ loss function:
  1. Rankings by average loss in the OOS period(s)
  2. Diebold-Mariano tests on average losses from these forecasts
  3. Goodness-of-fit tests

OOS forecast comparison results: Average loss

SP500, alpha=0.05. 1-factor GAS model, w/wo "hybrid" forcing variable, is best.

	SP500	DJIA	NIKKEI	FTSE
RW-125	0.914	0.864	1.290	0.959
RW-250	0.959	0.909	1.294	1.002
RW-500	1.023	0.976	1.318	1.056
GARCH-N	0.876	0.808	1.170	0.871
GARCH-Skt	0.866	0.796	1.168	*0.863*
GARCH-EDF	0.862	*0.796*	*1.166*	0.867
FZ-2F	*0.856*	0.798	1.206	1.098
FZ-1F	**0.853**	**0.784**	1.191	0.867
GARCH-FZ	0.862	0.797	1.167	0.866
Hybrid	0.869	0.797	**1.165**	**0.862**

OOS forecast comparison results : Diebold-Mariano t-stats

SP500, alpha=0.05. FZ-1F beats all. Not signif better than GARCH-EDF/Skew t

- A positive entry indicates the Column model is better than the Row model

	RW125	*G-EDF*	*FZ-2F*	*FZ-1F*	*G-FZ*	*Hybrid*
RW125	–	2.900	2.978	3.978	3.020	2.967
RW250	2.580	3.730	3.799	4.701	3.921	4.110
RW500	4.260	4.937	5.168	5.893	5.125	5.450
G-N	-2.109	3.068	1.553	2.248	2.818	0.685
G-Skt	-2.693	2.103	0.889	1.475	1.232	-0.403
G-EDF	-2.900	–	0.599	1.157	0.024	-0.769
FZ-2F	-2.978	-0.599	–	0.582	-0.555	-0.580
FZ-1F	-3.912	-1.198	-0.582	–	-1.266	-1.978
G-FZ	-3.020	-0.024	0.555	1.266	–	-0.914
Hybrid	-3.276	0.045	0.580	1.978	0.914	–

Avg OOS forecast rankings across all alphas

The best model for each alpha is always one of the proposed new models

- Ranking models by OOS average loss, for different tail probabilities

	0.01	**0.025**	**0.05**	**0.10**
RW-125	8	7.75	7.75	8
RW-250	8.25	8.25	8.75	9
RW-500	9.5	9.5	9.75	10
G-N	5.25	5	6.25	3.75
G-Skt	3	*2.5*	3.5	4.75
G-EDF	*2.5*	**2.25**	*3.25*	3.25
FZ-2F	5.5	7.25	6.25	5.5
FZ-1F	7	4.25	**3**	*3*
G-FZ	**2**	**2.25**	3.5	5.75
Hybrid	4	6	**3**	**2**

Goodness-of-fit tests for VaR and ES

- Under correct specification of the models for VaR and ES, we have

$$\mathbb{E}_{t-1}\left[\begin{array}{c}\partial L_{FZ0}\left(Y_t, v_t, e_t; \alpha\right)/\partial v_t \\ \partial L_{FZ0}\left(Y_t, v_t, e_t; \alpha\right)/\partial e_t\end{array}\right] = 0 \Leftrightarrow \mathbb{E}_{t-1}\left[\begin{array}{c}\lambda_{v,t} \\ \lambda_{e,t}\end{array}\right] = 0$$

- $\lambda_{v,t}$ and $\lambda_{e,t}$ can thus be considered as "**generalized forecast errors**."
- To reduce the impact of heteroskedasticity, we consider standardized versions, which also have mean zero:

$$\begin{aligned}\lambda^s_{v,t} &\equiv \frac{\lambda_{v,t}}{v_t} = \mathbf{1}\left\{Y_t \leq v_t\right\} - \alpha \\ \lambda^s_{e,t} &\equiv \frac{\lambda_{e,t}}{e_t} = \frac{1}{\alpha}\mathbf{1}\left\{Y_t \leq v_t\right\}\frac{Y_t}{e_t} - 1\end{aligned}$$

- We adopt the "dynamic quantile" **regression-based test** of Engle and Manganelli (2004) for VaR, and propose its natural analog for ES:

$$\begin{aligned}\lambda^s_{v,t} &= a_0 + a_1\lambda^s_{v,t-1} + a_2 v_t + \varepsilon_{v,t} \\ \lambda^s_{e,t} &= b_0 + b_1\lambda^s_{e,t-1} + b_2 e_t + \varepsilon_{e,t}\end{aligned}$$

OOS goodness-of-fit tests: VaR and ES

alpha=0.05. FZ-1F performs best

	GoF p-values: VaR				*GoF p-values: ES*			
	S&P	DJIA	NIK	FTSE	S&P	DJIA	NIK	FTSE
RW-125	0.021	0.013	0.000	0.000	0.029	0.018	0.006	0.000
RW-250	0.001	0.001	0.007	0.000	0.043	0.014	0.018	0.002
RW-500	0.001	0.001	0.000	0.000	0.012	0.011	0.001	0.000
GCH-N	0.031	0.139	0.532	0.000	0.001	0.006	0.187	0.000
GCH-Skt	0.003	0.085	0.114	0.000	0.003	0.085	0.282	0.000
GCH-EDF	0.003	0.029	0.583	0.000	0.014	0.098	0.527	0.000
FZ-2F	0.000	0.000	0.258	0.000	0.061	0.195	0.247	0.000
FZ-1F	0.242	0.248	0.317	0.019	0.313	0.130	0.612	0.003
GCH-FZ	0.005	0.001	0.331	0.000	0.018	0.011	0.389	0.000
Hybrid	0.001	0.069	0.326	0.000	0.010	0.159	0.518	0.000

Summary and conclusions

- The new Basel Accord will generate demand for models for Expected Shortfall
  - Existing models for volatility and VaR do not seem to do well for ES
- We exploit a recent result from decision theory that shows that ES is jointly elicitable with VaR
  - The "Fissler-Ziegel" loss function
- We propose new models and adaptations of old models, for forecasting ES
  - For $\alpha = 0.01$ and 0.025, the best models are GARCH estimated via FZ loss minimization and GARCH with nonparametric residuals.
  - For $\alpha = 0.05$ and 0.10, the best models are the one-factor GAS model, and the hybrid one-factor GAS/GARCH model.

Appendix

Basel Committee on Banking Supervision

Consultative document: A revised market risk framework, October 2013

- "The financial crisis exposed material weaknesses in the overall design of the framework for capitalising trading activities."
- "A number of weaknesses have been identified with using Value-at-Risk for determining regulatory capital requirements, including its inability to capture 'tail risk.' For this reason, the Committee proposed in May 2012 to replace Value-at-Risk with Expected Shortfall."
- "Risk reporting: the desk must produce, at least once a week... risk measure reports, including desk VaR/ES, desk VaR/ES sensitivities to risk factors, backtesting and p-value."

$\Rightarrow$ Expected shortfall is going to become an important part of risk management, complementing past emphasis on VaR.

Joint estimation of VaR and Expected Shortfall

- Fissler and Ziegel (2016, AoS) show that while ES is not elicitable, it is **jointly elicitable** with VaR, using the following class of loss functions:

$$
\begin{aligned}
L(Y, v, e; \alpha) \quad = \quad & (\mathbf{1}\{Y \leq v\} - \alpha)\left(G_1(v) - G_1(Y) + \frac{1}{\alpha} G_2(e)\, v\right) \\
& -G_2(e)\left(\frac{1}{\alpha}\mathbf{1}\{Y \leq v\}\, Y - e\right) - \mathcal{G}_2(e)
\end{aligned}
$$

  where

  - G_1 is weakly increasing
  - G_2 is strictly positive and increasing, and $\mathcal{G}_2' = G_2$.

- Minimizing this loss function yields VaR and ES:

$$
[VaR_t, ES_t] = \arg\min_{(v,e)} \; \mathbb{E}_{t-1}\left[L\left(Y_t, v, e; \alpha\right)\right]
$$

Expected Shortfall and VaR in location-scale models

- For intuition, assume that returns follow a conditional location-scale model (eg, ARMA-GARCH)

$$Y_t = \mu_t + \sigma_t \eta_t, \quad \eta_t \sim iid\ F_\eta (0, 1)$$

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- In this case, we have

$$\begin{aligned} VaR_t &= \mu_t + a\sigma_t, \quad \text{where} \quad a = F_\eta^{-1}(\alpha) \\ ES_t &= \mu_t + b\sigma_t, \quad \text{where} \quad b = \mathbb{E}[\eta_t | \eta_t \le a] \end{aligned}$$

and we we can recover (μ_t, σ_t) from (VaR_t, ES_t).

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- If $\sigma_t = \bar{\sigma} \ \forall \ t$, then $ES_t = c + VaR_t$, where $c = (b - a)\bar{\sigma}$

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- If $\sigma_t = \bar{\sigma}\ \forall\ t$, then $ES_t = c + VaR_t$, where $c = (b-a)\bar{\sigma}$
- If $\mu_t = 0\ \forall\ t$, then $ES_t = d \times VaR_t$, where $d = b/a$

Location-scale restrictions on the GAS model

- Baseline specification:

$$\begin{bmatrix} v_{t+1} \\ e_{t+1} \end{bmatrix} = \mathbf{w} + \mathbf{B} \begin{bmatrix} v_t \\ e_t \end{bmatrix} + \mathbf{A} \begin{bmatrix} \lambda_{v,t} \\ \lambda_{e,t} \end{bmatrix}$$

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- Motivated by the familiarity of **location-scale models**, where $Y_t = \mu_t + \sigma_t \eta_t$, we consider the following versions of this model

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1. $\mu_t = 0 \; \forall \; t$. This implies:

$$H_0 : \frac{w_e}{w_v} = \frac{a_{ev}}{a_{vv}} = \frac{a_{ee}}{a_{ve}} \cap b_e = b_v$$

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$$H_0 : \frac{a_{ev}}{a_{vv}} = \frac{a_{ee}}{a_{ve}} \cap b_e = b_v$$

3. $\sigma_t = \bar{\sigma} \; \forall \; t$. This implies:

$$H_0 : a_{ev} = a_{vv} \cap a_{ee} = a_{ve} \cap b_e = b_v$$

Outline

1. Motivation and introduction
2. Estimating Expected Shortfall (and Value-at-Risk)
   - The Fissler-Ziegel loss function
   - Dynamic models for VaR and ES
3. **Inference methods**
   - Assumptions and main results
   - Simulation study of finite-sample properties
4. Results for four international equity indices
   - In-sample parameter estimates and hypothesis tests
   - Out-of-sample forecast comparisons
5. Summary and conclusion

Statistical inference on models for ES and VaR

- The models we consider fit in the general framework of M-estimation for time series models:

$$\hat{\boldsymbol{\theta}}_T = \arg\min_{\boldsymbol{\theta}} \frac{1}{T}\sum\nolimits_{t=1}^{T} L\left(Y_t, v\left(\mathbf{Z}_{t-1}; \boldsymbol{\theta}\right), e\left(\mathbf{Z}_{t-1}; \boldsymbol{\theta}\right); \alpha\right)$$

- Our loss function is **non-differentiable**, but if we assume that Y_t is continuously distributed, this is easily handled.

- Under some regularity conditions, we obtain consistency and asymptotic Normality:

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^*\right) \xrightarrow{d} N\left(0, \mathbf{H}^{-1}\mathbf{G}\mathbf{H}^{-1}\right)$$

  - **G** is the usual covariance matrix of the scores (easy to estimate)
  - **H** is the Hessian, which is a bit trickier to obtain

Consistency

- **Assumption 1**: See paper for details. Key parts of this assumption:
  - Need finite first moments (unlike VaR estimation)
  - Need unique α-quantiles (see Zwingmann and Holzmann (2016) for results when this condition is violated).
- **Theorem 1:** Under Assumption 1, $\hat{\boldsymbol{\theta}}_T \xrightarrow{p} \boldsymbol{\theta}^0$ as $T \to \infty$.
- Proof is straightforward given Theorem 2.1 of Newey and McFadden (1994) and Corollary 5.5 of Fissler and Ziegel (2016).

Asymptotic normality

- **Assumption 2:** See paper for details. Key parts of this assumption:
  - Need $2+\delta$ moments of returns
- **Theorem 2:** Under Assumptions 1 and 2, we have

$$\sqrt{T}\mathbf{A}_T^{-1/2}\mathbf{D}_T(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0) \xrightarrow{d} N(0, I) \text{ as } T \to \infty$$

where

$$\begin{aligned}
\mathbf{A}_T &= \mathbb{E}\left[T^{-1}\sum_{t=1}^{T} g_t(\boldsymbol{\theta}^0)g_t(\boldsymbol{\theta}^0)'\right], \quad g_t(\boldsymbol{\theta}^0) = \frac{\partial L\left(y_t, v_t\left(\boldsymbol{\theta}^0\right), e_t\left(\boldsymbol{\theta}^0\right); \alpha\right)}{\partial \boldsymbol{\theta}} \\
\mathbf{D}_T &= \mathbb{E}\left[T^{-1}\sum_{t=1}^{T}\left\{\nabla' v_t(\boldsymbol{\theta}^0)\frac{f_t\left(v_t(\boldsymbol{\theta}^0)\right)}{-e_t(\boldsymbol{\theta}^0)\alpha}\nabla v_t(\boldsymbol{\theta}^0) + \frac{\nabla' e_t(\boldsymbol{\theta}^0)\nabla e_t(\boldsymbol{\theta}^0)}{e_t(\boldsymbol{\theta}^0)^2}\right\}\right]
\end{aligned}$$

- The proof builds on Huber (1967), Weiss (1991), Engle-Manganelli (2004).

Estimation of the asymptotic covariance matrix

- **Assumption 3:** See paper for details. Key parts of this assumption:
  - Bandwidth (c_T) satisfies $c_T \to 0$ and $c_T\sqrt{T} \to \infty$.
- **Theorem 3:** Under Ass'ns 1–3, $\hat{\mathbf{A}}_T - \mathbf{A}_T \xrightarrow{p} \mathbf{0}$ and $\hat{\mathbf{D}}_T - \mathbf{D}_T \xrightarrow{p} \mathbf{0}$, where

$$\begin{aligned}
\hat{\mathbf{A}}_T &= T^{-1}\sum_{t=1}^{T} g_t(\hat{\boldsymbol{\theta}}_T) g_t(\hat{\boldsymbol{\theta}}_T)' \\
\hat{\mathbf{D}}_T &= T^{-1}\sum_{t=1}^{T} \left\{ \frac{1}{2\hat{c}_T}\mathbf{1}\left\{\left|y_t - v_t\left(\hat{\boldsymbol{\theta}}_T\right)\right| < \hat{c}_T\right\} \frac{\nabla' v_t\left(\hat{\boldsymbol{\theta}}_T\right)\nabla v_t\left(\hat{\boldsymbol{\theta}}_T\right)}{-e_t\left(\hat{\boldsymbol{\theta}}_T\right)\alpha} \right. \\
&\qquad \left. + \frac{\nabla' e_t\left(\hat{\boldsymbol{\theta}}_T\right)\nabla e_t\left(\hat{\boldsymbol{\theta}}_T\right)}{e_t\left(\hat{\boldsymbol{\theta}}_T\right)^2} \right\}
\end{aligned}$$

- This extends Engle and Manganelli (2004) from dynamic VaR models to dynamic joint models for VaR and ES.

Simulation study

- For comparability with the existing literature, we simulate a GARCH process:

$$\begin{aligned} Y_t &= \sigma_t \eta_t \\ \eta_t &\sim iid\ F_\eta(0,1) \\ \sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma Y_{t-1}^2 \\ [v_t, e_t] &= [a, b]\, \sigma_t \end{aligned}$$

- $[\omega, \beta, \alpha] = [0.05, 0.9, 0.05]$.
- $F_\eta \in \{\ N(0,1)\ ,\ Skewt(5, -0.5)\ \}$.
- $\alpha \in \{\ 0.01\ ,\ 0.025\ ,\ 0.05\ ,\ 0.1\ , 0.2\}$.
- For std errors, we use $c_T = T^{-1/3}$.
- $T \in \{\ 2500\ ,\ 5000\ \}$, and $reps = 1000$.

Finite-sample properties of the estimator

Estimator is approximately unbiased, and 95% confidence intervals have reasonable coverage

Normal innovations, $\alpha = 0.05$

	$T = 2500$				$T = 5000$			
	β	γ	b_α	c_α	β	γ	b_α	c_α
True	0.900	0.050	-2.063	0.797	0.900	0.050	-2.063	0.797
Median	0.901	0.048	-2.051	0.800	0.899	0.049	-2.094	0.799
Bias	-0.013	0.005	-0.097	0.002	-0.008	0.002	-0.081	0.001
St dev	0.062	0.046	0.707	0.015	0.041	0.021	0.511	0.010
Cov'age	0.913	0.874	0.916	0.947	0.923	0.907	0.927	0.948

Finite-sample properties of the estimator

Std dev goes up for skew t errors, coverage remains reasonable

T=5000, $\alpha = 0.05$

	Normal				*Skew t*			
	β	γ	b_α	c_α	β	γ	b_α	c_α
True	0.900	0.050	-2.063	0.797	0.900	0.050	-2.767	0.651
Median	0.899	0.049	-2.094	0.799	0.898	0.048	-2.795	0.654
Bias	-0.008	0.002	-0.081	0.001	-0.011	0.003	-0.114	0.003
St dev	0.041	0.021	0.511	0.010	0.053	0.025	0.782	0.017
Cov'age	0.923	0.907	0.927	0.948	0.916	0.904	0.922	0.951

Estimation of VaR and ES

FZ estimation dominates CAViaR, but QMLE performs best here

Skew t innovations, T = 5000

	VaR			**ES**		
	MAE	*MAE ratio*		*MAE*	*MAE ratio*	
α	QML	CAViaR	FZ	QML	CAViaR	FZ
0.01	0.138	1.369	1.375	0.245	1.256	1.248
0.025	0.087	1.245	1.234	0.145	1.197	1.185
0.05	0.061	1.184	1.143	0.101	1.164	1.119
0.10	0.041	1.155	1.067	0.071	1.158	1.069
0.20	0.024	1.316	1.066	0.048	1.409	1.089

Finite-sample properties of the estimator

Std dev higher for smaller alpha, and coverage worse for smaller alpha

T=5000, Normal

	$\alpha = 0.01$				$\alpha = 0.10$			
	β	γ	b_α	c_α	β	γ	b_α	c_α
True	0.900	0.050	-2.665	0.873	0.900	0.050	-1.755	0.730
Median	0.899	0.049	-2.671	0.877	0.898	0.048	-1.778	0.730
Bias	-0.011	0.006	-0.089	0.004	-0.009	0.001	-0.072	0.000
St dev	0.049	0.033	0.805	0.015	0.040	0.020	0.435	0.009
Cov'age	0.884	0.876	0.888	0.937	0.922	0.902	0.934	0.960

For comparison: Finite-sample properties of QMLE

Estimator is approximately unbiased, and 95% confidence intervals have reasonable coverage

Skew t innovations

	$T = 2500$			$T = 5000$		
	ω	β	γ	ω	β	γ
True	0.500	0.950	0.500	0.500	0.950	0.500
Median	0.052	0.895	0.049	0.052	0.897	0.050
Bias	0.017	-0.023	0.005	0.006	-0.008	0.002
St dev	0.077	0.095	0.028	0.026	0.037	0.017
Cov'age	0.899	0.907	0.897	0.913	0.907	0.903

OOS forecast comparison results : Diebold-Mariano t-stats

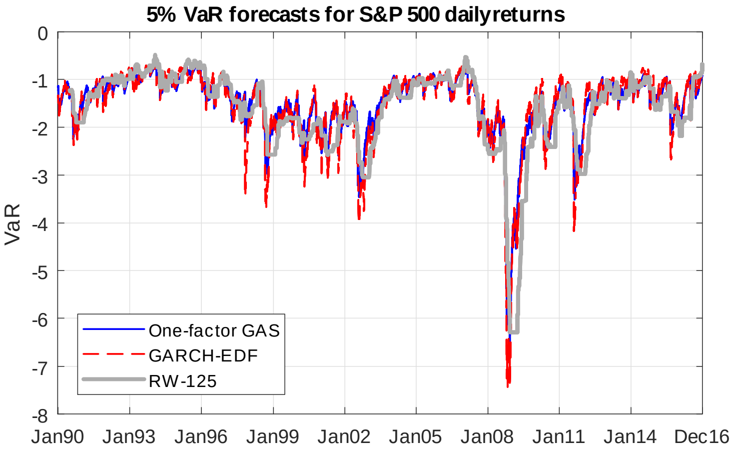
S&P 500 returns, alpha=0.025. G-FZ beats all, not signif better than G-EDF.

- A positive entry indicates the Column model is better than the Row model

	RW125	G-EDF	FZ-2F	FZ-1F	G-FZ	Hybrid
RW125	–	3.125	1.972	3.599	3.212	2.642
RW250	2.035	3.472	2.637	4.240	3.613	3.447
RW500	3.587	4.731	3.966	5.605	4.879	4.968
G-N	-1.100	3.522	1.645	2.346	3.835	1.963
G-Skt	-2.728	2.393	0.093	0.738	2.850	-0.447
G-EDF	-3.125	–	-0.595	-0.198	1.482	-1.500
FZ-2F	-1.972	0.595	–	0.348	1.111	0.368
FZ-1F	-3.599	0.198	-0.348	–	0.739	-1.406
G-FZ	-3.212	-1.482	-1.111	-0.739	–	-2.300
Hybrid	-2.642	1.500	-0.368	1.406	2.300	–

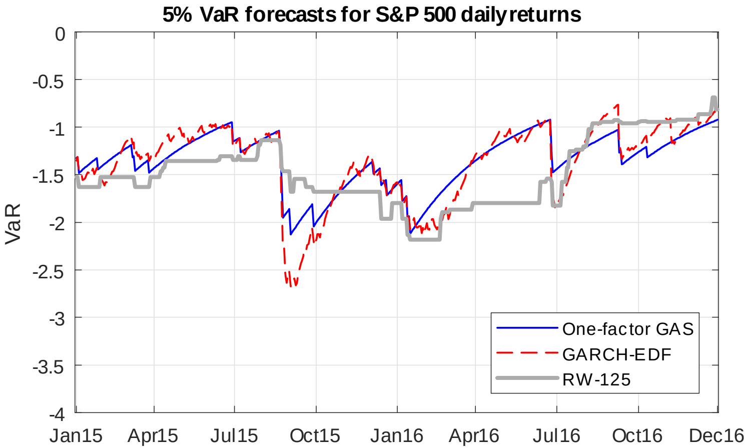
Dynamic Value-at-Risk: 1990-2016

VaR ranges from around -1% in mid 90s, to -6% in financial crisis



Dynamic Value-at-Risk: 2015-2016

The difference between the GAS and GARCH forcing variables is apparent here



OOS forecast rankings across various alphas: a=0.01

GARCH estimated by FZ loss is best on average

- Ranking models by OOS average loss, for different tail probabilities

	S&P	**DJIA**	**NIK**	**FTSE**	**Avg**
RW-125	7	8	10	7	8
RW-250	8	9	8	8	8.25
RW-500	10	10	9	9	9.5
G-N	6	6	5	4	5.25
G-Skt	5	3	*2*	*2*	3
G-EDF	4	*2*	3	**1**	*2.5*
FZ-2F	**1**	4	7	10	5.5
FZ-1F	9	7	6	6	7
G-FZ	3	**1**	**1**	3	**2**
Hybrid	*2*	5	4	5	4

OOS forecast rankings across various alphas: a=0.025

GARCH-EDF and GARCH-FZ are best on average

- Ranking models by OOS average loss, for different tail probabilities

	S&P	DJIA	NIK	FTSE	Avg
RW-125	8	8	8	7	7.75
RW-250	9	9	7	8	8.25
RW-500	10	10	9	9	9.5
G-N	7	6	4	3	5
G-Skt	5	3	**1**	**1**	*2.5*
G-EDF	*2*	*2*	3	*2*	**2.25**
FZ-2F	4	5	10	10	7.25
FZ-1F	3	4	6	4	4.25
G-FZ	**1**	**1**	*2*	5	**2.25**
Hybrid	6	7	5	6	6

OOS forecast rankings across various alphas: a=0.05

FZ-1F, with and without "hybrid" term, is best

- Ranking models by OOS average loss, for different tail probabilities

	S&P	DJIA	NIK	FTSE	Avg
RW-125	8	8	8	7	7.75
RW-250	9	9	9	8	8.75
RW-500	10	10	10	9	9.75
G-N	7	7	5	6	6.25
G-Skt	5	3	4	*2*	3.5
G-EDF	4	*2*	*2*	5	*3.25*
FZ-2F	*2*	6	7	10	6.25
FZ-1F	**1**	**1**	6	4	**3**
G-FZ	3	5	3	3	3.5
Hybrid	6	4	**1**	**1**	**3**

OOS forecast rankings across various alphas: a=0.10

FZ-1F with "hybrid" term is best

- Ranking models by OOS average loss, for different tail probabilities

	S&P	DJIA	NIK	FTSE	Avg
RW-125	8	8	8	8	8
RW-250	9	9	9	9	9
RW-500	10	10	10	10	10
G-N	3	*2*	5	5	3.75
G-Skt	7	4	4	4	4.75
G-EDF	4	3	3	3	3.25
FZ-2F	*2*	6	7	7	5.5
FZ-1F	**1**	7	*2*	*2*	*3*
G-FZ	6	5	6	6	5.75
Hybrid	5	**1**	**1**	**1**	**2**

OOS goodness-of-fit tests: VaR and ES

alpha=0.025. GARCH-EDF and FZ-1F performs best

	GoF p-values: VaR				*GoF p-values: ES*			
	S&P	DJIA	NIK	FTSE	S&P	DJIA	NIK	FTSE
RW-125	0.022	0.003	0.000	0.000	0.009	0.004	0.001	0.001
RW-250	0.005	0.007	0.002	0.000	0.023	0.039	0.010	0.005
RW-500	0.001	0.000	0.004	0.000	0.019	0.011	0.007	0.000
GCH-N	0.000	0.002	0.172	0.000	0.000	0.000	0.048	0.000
GCH-Skt	0.005	0.057	0.789	0.000	0.010	0.076	0.736	0.001
GCH-EDF	0.164	0.149	0.789	0.000	0.237	0.379	0.588	0.000
FZ-2F	0.000	0.117	0.000	0.000	0.001	0.341	0.000	0.000
FZ-1F	0.343	0.314	0.043	0.028	0.393	0.334	0.047	0.045
GCH-FZ	0.095	0.358	0.608	0.000	0.188	0.419	0.473	0.000
Hybrid	0.002	0.082	0.700	0.000	0.007	0.064	0.629	0.000