How far can we forecast?
Statistical tests of the predictive content

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The null hypothesis

• Assume that $Y_t$ is stationary and ergodic
• Let $\hat{Y}_{t+h|t}$ denote the forecast based on the information set $I_t$
• The forecast is uninformative if

$$H_0 : \quad \text{var}(Y_{t+h} - \hat{Y}_{t+h|t}) = \text{var}(Y_{t+h} - \mu)$$

$$e_{t+h|t}$$ $u_{t+h}$

• Since

$$\mathbb{E}(e_{t+h|t}^2) = \mathbb{E}[(Y_{t+h} - \mu) - (\hat{Y}_{t+h|t} - \mu)]^2$$

$\Rightarrow$ sufficient (but not necessary) condition for an uninformative forecast is $\hat{Y}_{t+h|t} = \mu$

• For rational forecasts with $\mathbb{E}(e_{t+h|t} \hat{Y}_{t+h|t}) = 0$ it follows that

$$\mathbb{E}(Y_{t+h} - \mu)(\hat{Y}_{t+h|t} - \mu) = \mathbb{E}(e_{t+h|t} + \hat{Y}_{t+h|t} - \mu)(\hat{Y}_{t+h|t} - \mu)$$

$$= \mathbb{E}(\hat{Y}_{t+h|t} - \mu)^2$$

$\Rightarrow H_0$ is equivalent to $\text{cov}(Y_{t+h}, \hat{Y}_{t+h|t}) = 0$. 
• **Maximum forecast horizon**

There exists some $h^*$ such that

$$H_0 : \quad \text{var}(e_{t+h\mid t}) \geq \text{var}(u_{t+h}) \quad \text{for } h > h^*$$

$h^*$ is called the maximum forecast horizon

• **Sequential test** of $H_0$ for $h = 1, 2, \ldots, h_{\text{max}}$. Stop when $H_0$ is not rejected for first time. Previous horizon is $\hat{h}^*$.

• **Non-stationary variables:**

$$Y_{t+h\mid t} = Y_t + \sum_{s=1}^{h} \Delta \hat{Y}_{t+s\mid t}$$

$$e_{t+h\mid t} = \sum_{s=1}^{h} \Delta e_{t+s\mid t} = \sum_{s=1}^{h} (\Delta Y_{t+s} - \Delta \hat{Y}_{t+s\mid t})$$

⇒ Non-predictability of $Y_{t+h}$ equivalent to non-predictability of $\Delta Y_{t+s}$ for $s = 1, \ldots, h$
Earlier work

a) Theil’s (1958) inequality coefficient:

\[ U2(h) = \frac{\sqrt{\sum_{t=1}^{n} (Y_{t+h} - \hat{Y}_{t+h|t})^2}}{\sqrt{\sum_{t=1}^{n} (Y_{t+h} - Y_{0,t+h})^2}} \]

where \( Y_{t+h} \) denotes some “naive forecast” (typically “no-change forecast”) \( \Rightarrow \) forecast uninformative if \( U2(h) = 1 \)

b) Nelson (1976) or Granger-Newbold (1986) measure:

\[ R^2(h) = 1 - \frac{\text{var}(e_{t+h|t})}{\text{var}(Y_{t+h})} \]

c) Diebold-Kilian (2001) forecastability measure:

\[ Q(L, h, k) = 1 - \frac{\mathbb{E}[L(e_{t+h|t})]}{\mathbb{E}[L(e_{t+k|t})]} \text{ where } k > h \]
• Note that for (i) stationary variables and (ii) MSE as the loss function:

\[ \lim_{k \to \infty} Q(MSE, h, k) = R^2(h) \]

• Our approach is based on \( R^2(h) \) (resp. MSE DIFF)

• We propose tests for the limiting horizon \( h^* \) beyond which forecasts become uninformative

• Empirical work suggests that economic forecasts of macroeconomic key variables (output growth, inflation) are informative 2-6 quarters ahead (or even less)

• Our empirical application based on survey forecasts from Consensus Economics indicates a maximum forecast horizon of typically less than one year
### Maximum forecast horizons in quarters

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Note: Regions considered are USA, Euro area, Japan, Germany, UK, Italy, Canada, and France. Variables are growth rates of real GDP, CPI, and real private consumption, and 1st differences of interest rates. $h = 0$ refers to the nowcast.
Notation and assumptions

• Forecast results from replacing $\theta$ by some estimator $\hat{\theta}$ such that

$$\hat{Y}_{t+h|t} = Y_{t+h|t}$$

• The forecast evaluation may be based on three different schemes:

  - recursive: $\{-T + 1, -T + 2, \ldots, t\}$
  - rolling: $\{t - T + 1, t - T + 2, \ldots, t\}$
  - fixed: $\{t - T + 1, \ldots, 0\}$

• We assume that we only observe actual values and forecasts but do not know (i) the forecasting model and (ii) the data used for estimating the model

• Forecast errors:

$$e_{t+h|t} = Y_{t+h} - Y_{t+h|t}^\theta$$

$$\hat{e}_{t+h|t} = Y_{t+h} - \hat{Y}_{t+h|t}$$
**Assumption 1:** (Time series process for \( Y_t \))

Let \( Y_t = \mu + u_t \) with \( u_t = \phi(L) \varepsilon_t, \phi(L) = 1 + \phi_1 L + \phi_2 L^2 + \cdots \) is a lag polynomial with all roots outside the unit circle, \( \sum_{i=1}^{\infty} |\phi_i| < \infty \) and \( \varepsilon_t \) is an i.i.d. white noise process with \( \mathbb{E}(\varepsilon_t) = 0 \) and \( \mathbb{E}(\varepsilon_t^2) = \sigma_{\varepsilon}^2 \). Furthermore \( \mathbb{E}|\varepsilon_t|^{2+\delta} < \infty \) for some \( \delta > 0 \).

**Assumption 2:** (Properties of the forecast)

(i) **Under** \( H_0: u_{t+h} = Y_{t+h} - \mu \) is independent of the past estimation error \( \hat{\theta}_t - \theta, \hat{\theta}_{t-1} - \theta, \ldots \).

(ii) The parameters are estimated consistently with

\[
a) \quad \hat{\theta}_0 - \theta = O_p\left(T^{-1/2}\right), \quad b) \quad \hat{\theta}_t - \hat{\theta}_0 = O_p\left(\frac{\sqrt{t}}{T}\right)
\]

(iii) Let \( D_{t+h}(\theta) = \partial Y_{t+h|t}^\theta / \partial \theta \) and \( \overline{D}_h(\theta) = n^{-1} \sum_{t=1}^{n} D_{t+h}(\theta) \)

\[
\frac{1}{n} \sum_{t=1}^{n} (D_{t+h}(\theta) - \overline{D}_h)^2 \xrightarrow{p} D^2 \quad \text{with} \quad 0 < D^2 < \infty
\]

\( \mathbb{E}|D_{t+h}(\theta)u_{t+h}|^{2+\delta} < \infty \) for some \( \delta > 0 \) and all \( t \).
Diebold-Mariano-type test

Comparing model forecast with unconditional mean $\bar{Y}_h$:

\[
\text{loss diff. } \delta^h_t = \hat{e}^2_{t+h|t} - (Y_{t+h} - \bar{Y}_h)^2
\]

DM (1995) statistic:

\[
d_h = \frac{1}{\hat{\omega}_\delta \sqrt{n}} \sum_{t=1}^{n} \delta^h_t,
\]

where $\hat{\omega}_\delta^2$ denotes the estimated long-run variance of $\delta^h_t$.

**Theorem 1:** Asymptotic distribution of the DM statistic:

If $T \to \infty$, $n \to \infty$, $n/T \to 0$ we have

\[
d_h = \sqrt{n} \frac{|\bar{u}|}{2\hat{\omega}_u} + O_p \left( \frac{n}{T} \right) \overset{d}{\to} \frac{|z|}{2},
\]

where $z \sim \mathcal{N}(0, 1)$.

$\Rightarrow$ non-standard as under $H_0$ the forecasts are *nested*
Modified DM statistic

Theorem 1 suggests the 2 adjusted DM statistics

\[ 2d_h \overset{d}{\to} |\mathcal{N}(0, 1)| \]

\[ \tilde{d}_h = \frac{1}{\hat{\omega}_u^2} \sum_{t=1}^{n} \delta_t^h \overset{d}{\to} \chi^2_1 \]

where \( \hat{\omega}_u^2 \) is a consistent estimator for the long-run variance of \( u_t = y_t - \bar{y} \).

- 5% critical values are 0.0627 and 0.0039, resp. \( \Rightarrow \) large size distortions
- Under \( H_1 \) : \( 2d_h = O_p(\sqrt{n}) \) and \( \tilde{d}_h = O_p(n) \). Nevertheless the local power is identical.
- If the model-based forecast is biased, the tests become conservative
## Actual sizes for various $n/T$ combinations ($\alpha = 0.05$)

<table>
<thead>
<tr>
<th>$T$</th>
<th>$n = 25$</th>
<th>$\tilde{d}_1$</th>
<th>$n = 50$</th>
<th>$\tilde{d}_1$</th>
<th>$n = 100$</th>
<th>$\tilde{d}_1$</th>
<th>$n = 200$</th>
<th>$\tilde{d}_1$</th>
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<tbody>
<tr>
<td>50</td>
<td>0.089</td>
<td>0.094</td>
<td>0.066</td>
<td>0.070</td>
<td>0.044</td>
<td>0.047</td>
<td>0.027</td>
<td>0.029</td>
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<tr>
<td>100</td>
<td>0.105</td>
<td>0.110</td>
<td>0.089</td>
<td>0.093</td>
<td>0.065</td>
<td>0.069</td>
<td>0.043</td>
<td>0.046</td>
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<tr>
<td>200</td>
<td>0.114</td>
<td>0.121</td>
<td>0.105</td>
<td>0.111</td>
<td>0.088</td>
<td>0.093</td>
<td>0.065</td>
<td>0.069</td>
</tr>
<tr>
<td>500</td>
<td>0.116</td>
<td>0.123</td>
<td>0.115</td>
<td>0.122</td>
<td>0.106</td>
<td>0.112</td>
<td>0.094</td>
<td>0.099</td>
</tr>
<tr>
<td>1000</td>
<td>0.110</td>
<td>0.117</td>
<td>0.116</td>
<td>0.123</td>
<td>0.114</td>
<td>0.121</td>
<td>0.108</td>
<td>0.114</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.049</td>
<td>0.049</td>
<td>0.049</td>
<td>0.049</td>
<td>0.050</td>
<td>0.050</td>
<td>0.051</td>
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</tbody>
</table>

Note: For $T = \infty$ the test statistics are computed using the true parameter values. Results are based on 100,000 replications.
Encompassing test

- Modifications based on the following decomposition:

\[
\sum_{t=1}^{n} \delta_t^h = \sum_{t=1}^{n} \left[ Y_{t+h} - \bar{Y}_h - (\hat{Y}_{t+h|t} - \bar{Y}_h) \right]^2 - (Y_{t+h} - \bar{Y}_h)^2
\]

\[
= \sum_{t=1}^{n} (\hat{Y}_{t+h|t} - \bar{Y}_h)^2 - 2 \sum_{t=1}^{n} (Y_{t+h} - \bar{Y}_h)(\hat{Y}_{t+h|t} - \bar{Y}_h)
\]

- First term does not contribute to power
- Reject \( H_0 \) if correlation between \( Y_{t+h} \) and \( \hat{Y}_{t+h|t} \) is large
- LM-type test statistic:

\[
\varrho_h = \frac{1}{\sqrt{n \hat{\omega}_\xi}} \sum_{t=1}^{n} \xi_t^h
\]

where \( \xi_t^h = (Y_{t+h} - \bar{Y}_h)(\hat{Y}_{t+h|t} - \bar{Y}_h) \)
and $\hat{\omega}_\xi^2$ denotes the corresponding long-run variance

\[ \hat{\omega}_\xi^2 = \hat{\gamma}_0^\xi + 2 \sum_{j=1}^{k} w_j^k \hat{\gamma}_j^\xi \]

\[ \hat{\gamma}_j^\xi = \frac{1}{n} \sum_{t=j+1}^{n} \xi_t \xi_{t-j} . \]

- Note that this test is asymptotically equivalent to the Mincer-Zarnowitz regression:

\[ Y_{t+h} = \beta_{0,h} + \beta_{1,h} \hat{Y}_{t+h|t} + u_{t+h} \]

but with $H_0 : \beta_{1,h} = 0$ instead of $\beta_{1,h} = 1$

- Encompassing test:

\[ Y_{t+h} = \lambda \hat{Y}_{t+h|t} + (1 - \lambda) \overline{Y}_h + \nu_{t+h} \]

\[ Y_{t+h} - \overline{Y}_h = \lambda (\hat{Y}_{t+h|t} - \overline{Y}_h) + \nu_{t+h} \]

with $H_0 : \lambda = 0$
Theorem 2:

Under Assumption 1–2, a recursive forecasting scheme with $h > h^*$, $T \to \infty$, $n \to \infty$ and $n/T \to 0$ we have

$$\varrho_h \xrightarrow{d} \mathcal{N}(0, 1)$$

- This might look trivial but is not. In the proof we show that

$$\hat{Y}_{t+h|t} - \bar{Y}_h \approx (\hat{\theta}_0 - \theta)D_{t+h}(\theta)$$

and thus the regressor tends to zero as $\hat{\theta} \xrightarrow{p} \theta$

- Asymptotically, the test is equivalent to the regression

$$Y_{t+h} = \beta^*_0, h + \beta^*_1, h D_{t+h}(\hat{\theta}) + \eta_{t+h}$$
Local power

Assume that the target value is generated as

\[ Y_{t+1} = \mu + \left( \frac{c}{\sqrt{n}} \right) X_t + u_{t+1} \]

such that

- regression forecast error: \( u_{t+1} + O_p(n/\sqrt{T}) \)
- unconditional forecast error \( u_{t+1} - \bar{u} + (c/\sqrt{n})(X_t - \bar{X}) \)

**Theorem 4:**

*Under the sequence of alternatives \( \beta = c/\sqrt{n}, X_t \sim iid(0, \sigma_x^2) \), Assumptions 1 – 2 and \( n/\sqrt{T} \to 0 \) it follows that*

\[
\tilde{d}_1 \xrightarrow{d} z_1^2 - 2\lambda z_2 - \lambda^2 \\
\hat{\varrho}_1 \xrightarrow{d} sign(c)z_1^2 + \lambda
\]

where \( \lambda^2 = c^2\sigma_x^2/\sigma_u^2 \) is the signal-to-noise ratio and \( z_1 \) and \( z_2 \) are independent \( \mathcal{N}(0, 1) \)

\( \Rightarrow \) tests are NOT asymptotically equivalent
Figure 1: Local power curves

Note: Broken line: DM-type test. Solid line: encompassing test
### Cases considered for Monte Carlo simulations

<table>
<thead>
<tr>
<th>case</th>
<th>DGP</th>
<th>forecast model</th>
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<tbody>
<tr>
<td>$MA(1)-AR(1)$</td>
<td>$y_t = \varepsilon_t + 0.5\varepsilon_{t-1}$</td>
<td>$\hat{y}_{t+h} = \hat{\theta}_1^h + \hat{\theta}_2^h y_t$</td>
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<td>$MA(2)-AR(1)$</td>
<td>$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.3\varepsilon_{t-2}$</td>
<td>$\hat{y}_{t+h} = \hat{\theta}_1^h + \hat{\theta}_2^h y_t$</td>
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<td>$AR(1)-AR(1)$</td>
<td>$y_t = 0.8y_{t-1} + \varepsilon_t$</td>
<td>$\hat{y}_{t+h} = \hat{\theta}_1^h + \hat{\theta}_2^h y_t$</td>
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<td>multivar. 1</td>
<td>$y_t = 0.5x_{t-1} + \varepsilon_t$</td>
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<td>multivar. 2</td>
<td>$y_t = 0.5x_{t-1} + 0.3x_{t-2} + \varepsilon_t$</td>
<td>$\hat{y}_{t+h} = \hat{\theta}_1^h + \hat{\theta}_2^h x_t + \hat{\theta}<em>3^h x</em>{t-1}$</td>
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Results for case ‘MA(1)-AR(1)’

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<th>2</th>
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<tr>
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<td>0.90</td>
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<td>0.79</td>
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Note: Values displayed in category ‘rejections’ denote percentage of rejections for each horizon $h$, values displayed in category ‘$\hat{h}^*$’, denote percentage of cases in which $h$ is identified as maximum forecast horizon. Bold entries refer to the true $h^*$. If test rejects for all horizons, $\hat{h}^*$ is set equal to $h = 4$. 
# Results for (most) remaining cases

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Note: The DM-type test uses $\tilde{d}_h$, the encompassing test employs $\beta_{1,h}$. 
US - GDP Growth - Quarterly Forecasts in June and December 2016, and June 2017

(% change y-o-y)
— MSPE variance ratio, + encompassing test, ○ DM-type test
USA CPI y-o-y

Eurozone CPI y-o-y

Japan CPI y-o-y

Germany CPI y-o-y
Conclusions

• Test for predictability at horizon $h$
• Determine $h^*$: maximum forecast horizon
• Problem: comparison of nested forecasts
• DM-type test and encompassing test
• Encompassing test outperforms the DM-type test
• We found $h^*$ between 1 and 5 quarters for macroeconomic key variables
• Extension to any comparison of nested forecast comparisons?