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Exact tests and confidence sets for the tail coefficient of α-stable distributions

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Abstract

In this paper, using the Monte Carlo (MC) method we propose an estimation and (at the same time) a test procedure for the stability parameter of α-stable distributions. One powerful advantage of the MC method is that it provides an exact significance level for finite samples, whose distribution can be far different from that of asymptotic samples on which the level of confidence interval for estimates is usually based. Statistical theory for the MC method is given. A simulation study compares the efficiency of our estimate with the Hill estimate (Hill, 1975). Construction of significance level based on the MC method is exploited and the corresponding power function is also studied. An empirical application demonstrates an easy implementation of our estimation and test procedure. It turns out that our estimate can improve the efficiency of any estimator for α in terms of mean square error.

Zusammenfassung

In der vorliegenden Arbeit wird ein auf der Monte-Carlo-Methode basierendes Schätz- und Testverfahren für den Stabilitätsparameter von α-stabilen Verteilungen vorgeschlagen. Ein entscheidender Vorteil dieses Verfahrens liegt darin, dass es genauere Konfidenzintervalle für endliche Stichprobenumfänge angibt, die häufig von denen aus asymptotisch ermittelten Verteilungen abweichen. Die statistische Theorie für die Monte-Carlo-Methode wird abgeleitet. Anhand einer Simulationsuntersuchung wird die Effizienz von unserem Verfahren und dem Schätzverfahren von Hill (1975) verglichen. Es wird gezeigt, wie sich die Konfidenzintervalle durch die Monte-Carlo-Methode konstruieren lassen. Zudem werden die zugehörigen Gütefunktionen berechnet. Weiterhin zeigt ein empirisches Beispiel die Einfachheit der empirischen Implementierung unseres Verfahrens. Es wird deutlich, dass unser Verfahren die Effizienz beliebiger Schätzer für α im Sinne vom mittleren quadratischen Fehler verbessern kann.

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Exact tests and confidence sets for the tail coefficient of α-stable distributions[1]

1 Introduction

Since the influential works of Mandelbrot (1963), stable Paretian (for short, α-stable) distributions have often been considered to be a more realistic distribution for asset returns than the normal distribution, because asset returns are typically heavy–tailed and excessively peaked around zero — phenomena that can be captured by α-stable distributions with $\alpha < 2$.

For the empirical application of the α-stable distributional assumption, the most important task is to estimate the stability parameter, α, precisely because the statistical inference for estimations and hypothesis tests under the α-stable distributional assumption depends crucially on α. Many estimates for α have been proposed in the literature: Hill estimator (Hill, 1975), Pickands (Pickands, 1975) and Dekkers et al. (1989) are the mostly widely used. For a rough check, the quantile estimation of McCulloch (1986) may be also used. Some modifications are also considered by many authors: Mittnik et al. (1998) modify the Pickands estimator using high order approximations and Huisman et al. (2001) propose a weighted Hill estimator taking into account the trade-off of bias and variance of Hill estimator. The common drawback of all these estimators is that their performance suffers severely when the sample size is small and/or the true α approaches 2. Moreover, the standard deviation of the estimates can be given only based on the asymptotic distribution of the corresponding estimators, which can be far different from that for finite samples. The poor performance of the available estimators for α, especially for finite samples, is the one of the most important reasons why — despite its statistically promising properties — the use of the α-stable distribution as a distributional assumption for fairly heavy-tailed financial data is rather limited.

In this paper, using the Monte Carlo (MC) method we propose an estimation[2] and, at the same time, an exact test for the stability parameter of α-stable distributions. This is because it is possible construct an exact confidence interval in the estimation procedures. The advantages of our method are summarized in the following two points: with our method, the efficiency of any estimator for α can

[1]The views expressed in this paper are those of the author and not necessarily those of the Deutsche Bundesbank. Research support from the Alexander-von-Humboldt Foundation is gratefully acknowledged by the two authors. Jean-Marie Dufour, C.R.D.E. and Départment de Sciences Economique, Université de Montréal, Montréal, Québec H3C 3J7, Canada Jeong-Ryeol Kurz-Kim, Deutsche Bundesbank, Wilhelm-Epstein-Straße 14, 60431 Frankfurt am Main, Germany

[2]An estimation procedure which is based on the MC method is termed Hodges-Lehmann estimation in the literature.

be improved[3] and the MC method provides an exact confidence interval for finite samples.

The rest of the paper is organized as follows. Section 2 gives a brief summary of the statistical theory for the MC method and the α-stable distributions. The procedures for the estimation and the test are also described. In Section 3, the statistical property of the estimator and the power of the test are studied. Based on simulation study, a comparison with Hill estimator is also given. An empirical application is presented in Section 4. The empirical application clearly demonstrates the advantage of the estimation and test based on the MC tests. Section 5 summarizes the paper and contain some concluding remarks.

2 Estimation and test for the stability parameter

2.1 Theory of the Monte Carlo method

The technique of the MC method, originally proposed by Dwass (1957) for implementing permutation tests and later extended by Barnard (1963) and recently reconsidered by Dufour (2002), provides an attractive method of building exact tests from statistics whose finite sample distribution is intractable but can be simulated. The most promising advantage of the MC method is that, in contrast to bootstrap techniques and other conventional test methods which have only asymptotic justification, an exact finite-sample inference can be obtained. Consequently, the validity of this MC method based an exact randomized test does not depend on the number of replications made. For more details on the MC method, see Birnbaum (1974), and Dufour and Kiviet (1998). In line with the results in Dufour (2002), we summarize the MC method based test when the null distribution of a test statistic does not involve nuisance parameters.

Let $S_1, \ldots, S_N$ be random samples with independent and identically distributed ($i.i.d.$) real random variables ($r.v.s$) with the same distribution S. It is assumed that $S_1, \ldots, S_N$ are also independent. Suppose that the distribution of S under H_0 may not be easy to compute analytically but can be simulated. However, it turns out that the exchangeability of $S_1, \ldots, S_N$ is sufficient for most of the results presented below.[4] The methodology of Monte Carlo tests provides a simple way of allowing the theoretical distribution $F(x)$ to be replaced by its sample analogue based on $S_1, \ldots, S_N$ as

$$\hat{F}_N [x;\, S(N)] = \frac{1}{N}\sum_{i=1}^{N} \mathrm{I}_{[0,\infty)} (x - S_i) \tag{1}$$

[3]For the demonstration, we shall employ the Hill estimator as examples because of its popularity.

[4]The elements of a random vector $(S_1, S_2, \ldots, S_N)'$ are exchangeable if $(S_{r_1}, S_{r_2}, \ldots, S_{r_N})' \sim (S_1, S_2, \ldots, S_N)'$ for any permutation $(r_1, r_2, \ldots, r_N)$ of the integers $(1, 2, \ldots, N)$.

where $S(N) = (S_1, \ldots, S_N)'$, and $\mathrm{I}_A(x)$ is the usual indicator function associated with the set A, i.e. $\mathrm{I}_A(x) = 1$, when $x \in A$ and 0, otherwise. Furthermore, let

$$\hat{G}_N[x;\, S(N)] = \frac{1}{N}\sum_{i=1}^{N} \mathrm{I}_{[0,\infty)}(S_i - x)\ . \tag{2}$$

be the corresponding sample function of the tail area. The sample distribution function is related to the ranks $R_1, \ldots, R_N$ of the variables $S_1, \ldots, S_N$ (when placed in ascending order) by the expression:

$$R_j = N\hat{F}_N[S_j;\, S(N)] = \sum_{i=1}^{N} s(S_j - S_i)\ ,\ j = 1, \ldots, N\,. \tag{3}$$

The main idea behind the MC method is that critical values and/or compute p-values can be obtained by replacing the "theoretical" null distribution $F(x)$ through its simulation-based "estimate" $\hat{F}_N(x)$ in a way that will preserve the level of the test in *finite samples, irrespective of the number N of replications used* as follows. Let S_0 be an empirical sample of interest, $(S_1, \ldots, S_N)'$ a simulated $(N \times 1)$-, and consequently $(S_0, S_1, \ldots, S_N)'$ a $((N+1)\times 1)$-random vector of exchangeable real *r.v.s.*[5] Moreover, let $\hat{F}_N(x) \equiv \hat{F}_N[x; S(N)]$, $\hat{G}_N(x) = \hat{G}_N[x; S(N)]$ and $\hat{F}_N^{-1}(x)$ be defined as in (1) - (2), and set

$$\hat{p}_N(x) = \frac{N\hat{G}_N(x) + 1}{N+1}\,. \tag{4}$$

Then

$$\mathsf{P}\big[\hat{G}_N(S_0) \le \alpha_1\big] = \mathsf{P}\big[\hat{F}_N(S_0) \ge 1 - \alpha_1\big] = \frac{I[\alpha_1 N] + 1}{N+1}\ ,\ \text{for } 0 \le \alpha_1 \le 1\,, \tag{5}$$

$$\mathsf{P}\big[S_0 \ge \hat{F}_N^{-1}(1 - \alpha_1)\big] = \frac{I[\alpha_1 N] + 1}{N+1}\ ,\ \text{for } 0 < \alpha_1 < 1\,, \tag{6}$$

and

$$\mathsf{P}\big[\hat{p}_N(S_0) \le \alpha\big] = \frac{I[\alpha(N+1)]}{N+1}\ ,\ \text{for } 0 \le \alpha \le 1\,. \tag{7}$$

For practical purposes, α_1 and N will be chosen as

$$\alpha = \frac{I[\alpha_1 N] + 1}{N+1}, \tag{8}$$

which is the desired significance level. Provided N is reasonably large, α_1 will be very close to α; in particular, if $\alpha(N+1)$ is an integer, we can take

$$\alpha_1 = \alpha - \frac{(1-\alpha)}{N}\,,$$

[5] The zero probability of ties is assumed, but the results are still valid for a positive probability of ties.

in which case we see easily that the critical region $\hat{G}_N(S_0) \leq \alpha_1$ is equivalent to $\hat{G}_N(S_0) < \alpha$. For $0 < \alpha < 1$, the randomized critical region $S_0 \geq \hat{F}_N^{-1}(1-\alpha_1)$ has the same level (α) as the non-randomized critical region $S_0 \geq F^{-1}(1-\alpha)$ or, equivalently, the critical regions $\hat{p}_N(S_0) \leq \alpha$ and $\hat{G}_N(S_0) \leq \alpha_1$ have the same level as the critical region $G(S_0) \equiv 1 - F(S_0) \leq \alpha$. The validity of the p-values calculated for continuous distributions is proved in Lemma 2.1.1 in Dufour (2002).

2.2 Summary on α-stable distributions

An r.v. X is said to be stable if for any positive numbers A and B, there is a positive number C and a real number D so that $AX_1 + BX_2 \overset{d}{=} CX + D$, where X_1 and X_2 are independent r.v.s with $X_i \overset{d}{=} X, i = 1, 2$; and "$\overset{d}{=}$" denotes equality in distribution. Moreover, $C = (A^\alpha + B^\alpha)^{1/\alpha}$ for some $\alpha \in (0, 2]$, where the exponent α is called the index of stability. When $0 < \alpha < 2$, the tails of the distribution are thicker than those of the normal distribution. The tails become thicker as α decreases so that moments of order α or higher do not exist. A stable r.v., X, with index α is called α-stable. The α-stable distributions are described by four parameters denoted by $S(\alpha, \beta, \mu, \sigma)$. Although the α-stable laws are absolutely continuous, their densities can be expressed only by a complex special function except in some special cases.[6] Therefore, the logarithm of the characteristic function, $\phi(t)$, of the α-stable distribution is the best way of characterizing all members of this family and given that

$$\ln \phi(t) = \ln \int_{-\infty}^{\infty} e^{ist} d\mathrm{P}(S < s) = \begin{cases} -\sigma^\alpha |t|^\alpha [1 - i\beta \operatorname{sign}(t) \tan\frac{\pi\alpha}{2}] + i\mu t, & \text{for } \alpha \neq 1, \\ -\sigma |t| [1 + i\beta \frac{\pi}{2} \operatorname{sign}(t) \ln |t|] + i\mu t, & \text{for } \alpha = 1, \end{cases}$$

The shape of the α-stable distribution is determined by the stability parameter α. For $\alpha = 2$ the α-stable distribution reduces to the normal distribution, (i.e. $S(2, 0, \sigma, \mu) \equiv N(\mu, 2\sigma^2)$), the only member of the α-stable family with finite variance. If $\alpha < 2$, moments of order α or higher do not exist and the tails of the distribution become thicker, i.e. the magnitude and frequency of outliers (from the viewpoint of the Gaussian) increase as α decreases. Skewness is governed by $\beta \in [-1, 1]$. When $\beta = 0$, the distribution is symmetric. The location and scale of the α-stable distributions are denoted by μ and σ. The standardized version of the α-stable distribution is given by $S((x-\mu)/\sigma; \alpha, \beta, 1, 0)$.

Assume that random sequence $\{X_i\}_{i=1}^\infty$ is in the domain of attraction[7] (DA) of an α-stable law with index $\alpha \in (0, 2)$. This is equivalent to saying that $X_1, X_2, \cdots$

[6] The Gaussian ($\alpha = 2$), the symmetric Cauchy ($\alpha = 1, \beta = 0$) and the Lévy distribution ($\alpha = 0.5, \beta = 1$) are the special cases whose densities are expressible via elementary functions.

[7] See Samorodnitsky and Taqqu, 1994, p. 5 for a definition of domain of attraction.

are *i.i.d.* *r.v.s* and that constants $a_n > 0$ and $b_n \in \mathbf{R}$ exist, such that

$$b_n^{-1}(S_n - a_n) \xrightarrow{d} S_{(\alpha)}, \tag{9}$$

where $S_n = X_1 + \cdots + X_n$; $S_{(\alpha)}$ is an α-stable *r.v.*; and " $\xrightarrow{d}$ " denotes convergence in distribution. If the X_i's are α-stable, then (9) holds with $b_n = n^{\frac{1}{\alpha}}$, and

$$a_n = \begin{cases} \mu(n^{1-\frac{1}{\alpha}} - 1), & \text{for } \alpha \neq 1, \\ \frac{2}{\pi}\sigma \ln n, & \text{for } \alpha = 1, \end{cases} \tag{10}$$

and $b_n^{-1}(S_n - a_n) \stackrel{d}{=} X_1$. The assumption that the X_i's are in the DA of an α-stable law is more general than assuming that they are α-stable distributed, because the former requires only conditions on the *tails* of the distribution. DA condition (9) is equivalent to

$$\mathrm{P}(|X_1| > x) = x^{-\alpha}L(x), \quad x > 0, \tag{11}$$

where $L(x)$ is a slowly varying function.[8]

A strong argument in favor of the α-stable distribution for a distributional assumption for heavy-tailed empirical data is that only the α-stable distribution can serve as limiting distribution of sums of *i.i.d.* *r.v.s* as proved in Zolotarev (1986). For more details on the α-stable distributions and discussions of the role of the α-stable distribution in financial market and macroeconometric modeling, see Zolotarev (1986), Samorodnitsky and Taqqu (1994), McCulloch (1997), Rachev et al. (1999) and Rachev and Mittnik (2000).

2.3 Test statistic and test procedure

In this section, we introduce the test statistic and test procedure based on the MC method. For our random sample, some assumptions are needed and summarized as follows.

Assumption 1 X *is an i.i.d. symmetric* α*-stable distributed r.v. about* μ *with* $\alpha \in (0, 2]$*, i.e.* $X_1, X_2, \ldots, X_n \sim S(\alpha, 0, \mu, \sigma)$*, denoted as* $S_\alpha S$*.*

Note that X is not strictly α-stable r.v. Assumption 1 is something of generalization of the usual assumption of standard symmetric α-stable r.v. (i.e., $c = 1, \mu = 0$) because the centering parameter is also an important issue from the viewpoint of

[8] $L(x)$ is a slowly varying function as $x \to \infty$, if for every constant $c > 0$, $\lim_{x\to\infty} \frac{L(cx)}{L(x)}$ exists and is equal to 1. See Ibragimov and Linnik (1971, p. 394) for more details on slowly varying functions.

empirical relevance. On the other hand, Assumption (1) is rather restrictive from the empirical viewpoint because densities of many empirical data are asymmetric.[9]

Now, we test our random sample, S_0, for

$$H_0 : \alpha = \alpha_0. \tag{12}$$

To perform this test, we need a test statistic which is free from nuisance parameters under the null hypothesis. A possible statistic may be given as

$$ST = \hat{\alpha} - \alpha_0, \tag{13}$$

where $\hat{\alpha}$ may be any consistent estimator for α. The most popular estimation for α is the Hill estimator (Hill, 1975), which is a simple nonparametric estimator based on an order statistic. Owing to its simplicity and popularity, we use the Hill estimator for constructing our test statistic in (13). The Hill estimator is given as

$$\hat{\alpha}_H = \left[\left(k^{-1} \sum_{j=1}^{k} \ln X_{n+1-j:n} \right) - \ln X_{n-k:n} \right]^{-1}, \tag{14}$$

where k is a truncation parameter to be optimally chosen depending on the sample size, n, and tail–thickness, α, as $k = k(n, \alpha)$, and $X_{j:n}$ denotes the j-order statistic of the sample size n. The asymptotic properties of the Hill estimator have been studied by many authors and are now well developed: Mason (1982) and Hsing (1992) consider weak consistency of the Hill estimator for independent and dependent cases, respectively. The strong consistency is proved by Deheuvels et al. (1988). Goldie and Smith (1987) provide asymptotic normality of the Hill estimator.

Owing to the bias-variance trade-off of the Hill estimator, for a comparison with other competing estimators in terms of asymptotic mean square error, the asymptotic bias caused by the second-order behavior of the slowly varying function must be also taken into account. According to Theorem 1 in de Haan and Peng (1998), the distribution of the Hill estimator is given as when $k = k(n) \to \infty$, $k/n \to 0$ as $n \to \infty$

$$\sqrt{k} \left[\frac{1}{\hat{\alpha}_H} - \frac{1}{\alpha} \right] \longrightarrow_{k\to\infty} N \left(\frac{\alpha^3 \lambda}{\rho - \alpha}, \alpha^{-2} \right),$$

where $\lambda \in \mathbb{R}$ equals $\lim_{n\to\infty} \sqrt{k}\alpha^2 aF(1-\frac{n}{k})$ mit a a measurable function of constant sign and the constant $\rho \in (-\infty, 0]$ is the second-order parameter governing the rate

[9] Nevertheless, it makes sense to consider estimating α under symmetric assumption from two reasons: for some empirical fields such as analysis of value-at-risk the possible asymmetry is irrelevant because we are only interested in one-side (lower) tail. The second reason is that for the range of α of empirical relevance, say, $\alpha > 1.5$, the neglection of possible asymmetry in the estimation of α is relatively non-severe.

of convergence. Because of the interdependency of n and α or, in other words, the bias-variance trade-off for the Hill estimator, the asymptotic mean square error — despite of the consistency of the Hill estimator — is not equal to the asymptotic variance and is given as

$$MSE(\hat{\alpha}_H) = k^{-1}\left(\alpha^2 + \frac{\alpha^6\lambda^2}{(\rho-\alpha)^2}\right), \tag{15}$$

For estimating the tail–thickness parameter via the Hill estimator, two practical problems should be solved. The first is how to choose the truncation parameter, k, and the other is how to choose the centering parameter from the empirical data.[10] Note that the Hill estimator is not location invariant. This means that X has to be centered properly in the beginning of the estimation. The theoretical relation between k and α may be also driven from the second-order property of the slowly varying function for symmetric case as

$$k = n\,\frac{\alpha-1}{\Gamma(2-a)\sin(\pi * a/2)}.$$

The obvious problem in using this relation is that k is again a function of α which has to be estimated.

The theoretical centering for variables in the DA of an α-stable law is as given in (10). Nevertheless, this again contains the unknown α. Usually, median or truncated mean is chosen for the centering parameter. Despite the existence of the first moment, the mean does not serve as a centering parameter because of its high fluctuation, especially, if α is small — say, smaller than 1.2. Based on the property of DA of an α-stable law, one theoretical appealing choice is surely $\ln X_{k:n}$.[11] However, this depends again on the choice of k. We take median as a suitable centering. To improve the efficiency of the Hill estimator, both sides of tail also have to be included in the estimation. Using the median as a centering parameter and taking both sides of tail into account, the Hill estimator in 14 may be re-written as

$$\hat{\alpha}_{HM} = \left[\left(k^{-1}\sum_{j=1}^{k}\ln|\tilde{X}_{n+1-j:n}|\right) - \ln|\tilde{X}_{n-k:n}|\right]^{-1}, \tag{16}$$

where $\tilde{X}_{j:n}$ stands for the j-th order statistic of the sample size n re-located by median. Consequently, the test statistic below is

$$ST = \hat{\alpha}_{HM} - \alpha_0. \tag{17}$$

To estimate the stability parameter using our Monte Carlo-based method, the test statistic in (13) should be nuisance-free. Because the estimator in (16) is location and scale-invariant, the test statistic (17) is pivotal as proved in the following lemma.

[10] Owing to the invariance of the Hill estimator with respect to scale, there is no need for any concern about re-scaling of empirical data, i.e. it may be an arbitrary one.

[11] When $\ln X_{k:n} < 1$ the proper dealing will be $\ln(1 + X_{k:n})$.

Proposition 1 *Let* $X_1, X_2, \ldots, X_n$ *be i.i.d. random variables which follow a* $S(\alpha, \beta, \mu, \sigma)$ *distribution, and let*

$$\hat{\alpha} = a(X_1, X_2, \ldots, X_n) \tag{18}$$

be an estimator of α. *If the estimator* $\hat{\alpha}$ *is scale invariant, i.e.*

$$\hat{\alpha} = a(cX_1, \ldots, cX_n) = a(X_1, X_2, \ldots, X_n), \text{ for all } c > 0\,, \tag{19}$$

then the estimator $\hat{\alpha}$ *has a distribution which only depends on* α, β *and* μ/σ. *If furthermore,* $\hat{\alpha}$ *is location-scale-invariant, i.e.*

$$\hat{\alpha} = a(cX_1+d, \ldots, cX_n+d) = a(X_1, X_2, \ldots, X_n)\ , \text{ for all } c > 0 \text{ and } d \in \mathbf{R}, \tag{20}$$

then the estimator $\hat{\alpha}$ *has a distribution which only depends on* β.

Proof 1 *To obtain the first result, we observe that*

$$X_i/\sigma \sim S(\alpha, \beta, \mu/\sigma, 1)\,,\ i = 1, \ldots, n\,. \tag{21}$$

Then, using the scale-invariance property (19) with $c = 1/\sigma$, *we may write:*

$$\hat{\alpha} = a(X_1/\sigma, \ldots, X_n/\sigma)\ , \tag{22}$$

from which we see that the distribution of $\hat{\alpha}$ *only depends on* α, β *and* μ/σ. *Similarly, under the location-scale invariance condition (20), the result follows on observing that*

$$(X_i - \mu)/\sigma \sim S(\alpha, \beta, 0, 1)\,,\ i = 1, \ldots, n\,, \tag{23}$$

hence, taking $c = 1/\sigma$ *and* $d = -d/\sigma$,

$$\hat{\alpha} = a(X_1^*, \ldots, X_n^*)\ , \tag{24}$$

where $X_i^* = (X_i - \mu)/\sigma,\ i = 1, \ldots, n$.

If we consider symmetric stable distributions ($\beta = 0$), the distribution of the scale-invariant estimator only depends on μ/σ. If, furthermore, the median is zero ($\mu = 0$), the distribution of $\hat{\alpha}$ only depends on α. Of course, the same remarks also apply to any function of $g(\hat{\alpha}, \alpha)$ of $\hat{\alpha}$ and α, such as $\hat{\alpha} - \alpha$. Similarly, if $\beta = 0$, the distribution of $\hat{\alpha}$ (or, more generally, of $g(\hat{\alpha}, \alpha)$) only depends on α, irrespective of the unknown values of μ and σ. Therefore, $g(\hat{\alpha}, \alpha)$ is a pivotal function.

2.4 Statistical property of the MC based estimator

In this section we discuss some statistical properties of the MC-based estimator and compare the performance of the estimator with the Hill estimator. The idea of the MC based estimator goes back to the work of Hodges and Lehmann (1963). The MC based estimation procedure is as follows:

1 Determine the set of possible α under null hypothesis. From the viewpoint of empirical relevance it will be reasonably $\alpha_0 \in (1\ 2]$.

2 Calculate test statistics $(\hat{\alpha} - \alpha_0)$ for every α_0, where the step length of two neighborhoods of α_0 may be 0.01, for example.

3 Generate typically 99 or 999 samples for every element of the set by a stable random variable generator and calculate test statistics.

4 Compute p-values under every possible null hypothesis.

5 Take the $\hat{\alpha}$ as the estimate for α at which the p-value has its minimum.

To compare the performances of the MC-based estimator with the (median-centered) Hill estimator, we perform simulations by drawing from symmetric α-stable pseudo-random variables. Specifically, we consider α-values of 1.2, 1.5 and 1.8 and a sample size of n= 250, 1,000 and 5,000. For each of the resulting 12 (α, n)-combinations we generated 1,000 replications of the corresponding random samples.

For the Hill estimator, we use the optimal k by assuming α is to be known and take k from Table in Mittnik et al. (1998) as before.[12] Table 1 shows the simulation results.

Table 1. RMSE ($\sqrt{BIAS}$, SD^a) of Hill and MC estimator

α	sample size / estimator	250	1000	5000
1.2	Hill	0.1626 (0.0986, 0.1293)	0.0874 (0.0548, 0.0680)	0.0342 (0.0212, 0.0269)
	MC	0.1369 (0.0836, 0.1084)	0.0735 (0.0457, 0.0575)	0.0279 (0.0166, 0.0224)
1.5	Hill	0.1744 (0.1112, 0.1344)	0.1003 (0.0632, 0.0778)	0.0440 (0.0276, 0.0343)
	MC	0.1357 (0.0857, 0.1052)	0.0761 (0.0477, 0.0593)	0.0327 (0.0201, 0.0258)
1.8	Hill	0.1944 (0.1241, 0.1497)	0.0865 (0.0528, 0.0685)	0.0525 (0.0338, 0.0402)
	MC	0.1645 (0.1031, 0.1282)	0.0744 (0.0454, 0.0589)	0.0435 (0.0278, 0.0335)

^{a}Standard deviation.

[12] This is an ideal pre-condition which is never given in empirical works and, hence, the Hill estimate is expected to show its best efficiency.

As is expected, the MC-method based estimator shows a smaller MSE than that of the Hill estimator for all considered α and sample size. The efficiency improvement, i.e. the ratio between the MSE of the Hill estimator and the MSE of the MC method based estimator, is approximately 75%–85%.

2.5 Power function of the Monte Carlo method based test

The theoretical size and power of the MC-method-based test is considered in Dufour (2002). Although the discrepancy of the correct size and the superior power of the MC method based test over the conventional test goes to zero as sample size goes to infinity, the behavior of the power function for the finite sample is usually of interest.

To check the power of our MC-method-based test, we perform a simulation study by drawing from symmetric α-stable pseudo-$r.v.s$[13] re-located by the median. As pseudo-empirical data, we take the same α-stable random sample as generated above, and test $H_0 : \alpha = \alpha_0$, where α_0 is assumed to be values from 1.0 to 2.0 in steps of 0.1. For sample size $n = 100$, 250, 500, 1,000, 2,000, 5,000 and 10,000 are selected and the number of replication is 10,000. For calculating the test statistic in (13) containing the Hill estimator, we use as optimal k the numbers tabulated in Mittnik et al. (1998). To demonstrate the power function, we select a usual significance level of 95%. Figures 1 shows the power functions for the selected α, n and percentage points as described above.

As is expected, the power converges to the corresponding ideal value for each given significance level, as the sample size grows. A sample size of 2,000 gives a rather satisfactory power. A large loss of power can be observed in extremely small sample sizes.

3 An empirical application

To illustrate the use of the Monte Carlo test, we employ some empirical data. They are a daily return of S$P500 (7,421 observations, July 2, 1962–Dec. 28, 1992), German Stock Index (8,922 observations, Jan. 4, 1960–Sep. 9, 1995), exchange rates of the Japanese yen against US$ (5,159 observations, Jan. 8, 1973–July 28, 1994) and exchange rates of Deutsche Mark against US$ (5,159 observations, Jan. 8, 1973–July 28, 1994). Figure 2 shows the empirical densities of the four time series (thick solid line) compared with the normal density (thin solid line).

[13]The pseudo α-stable $r.v.s$ were generated with the algorithm of Chambers et al. (1976) improved by Weron (1996), as implemented in Matlab Windows V. 5.3.

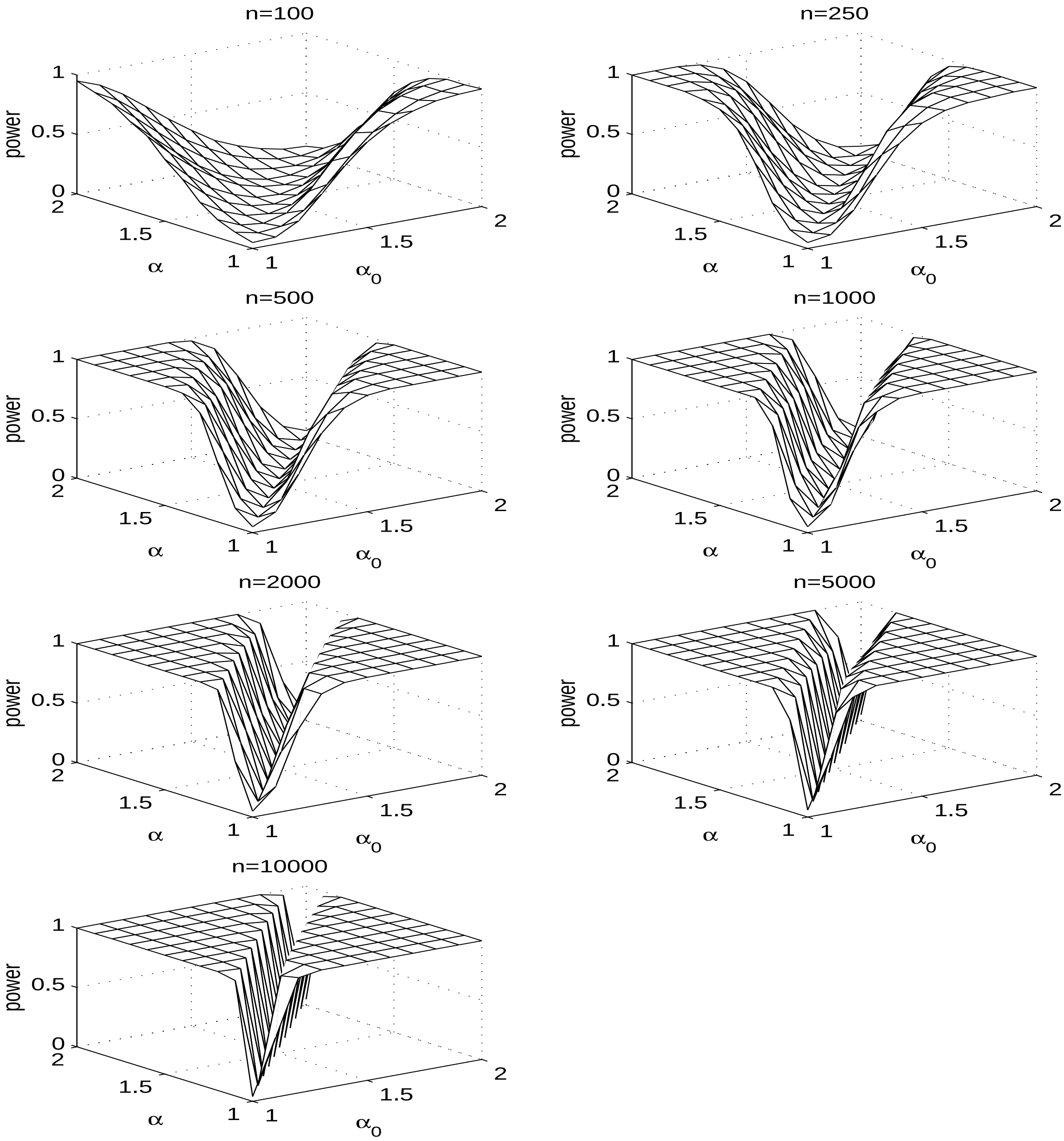

Figure 1: Power functions for 95% significance level

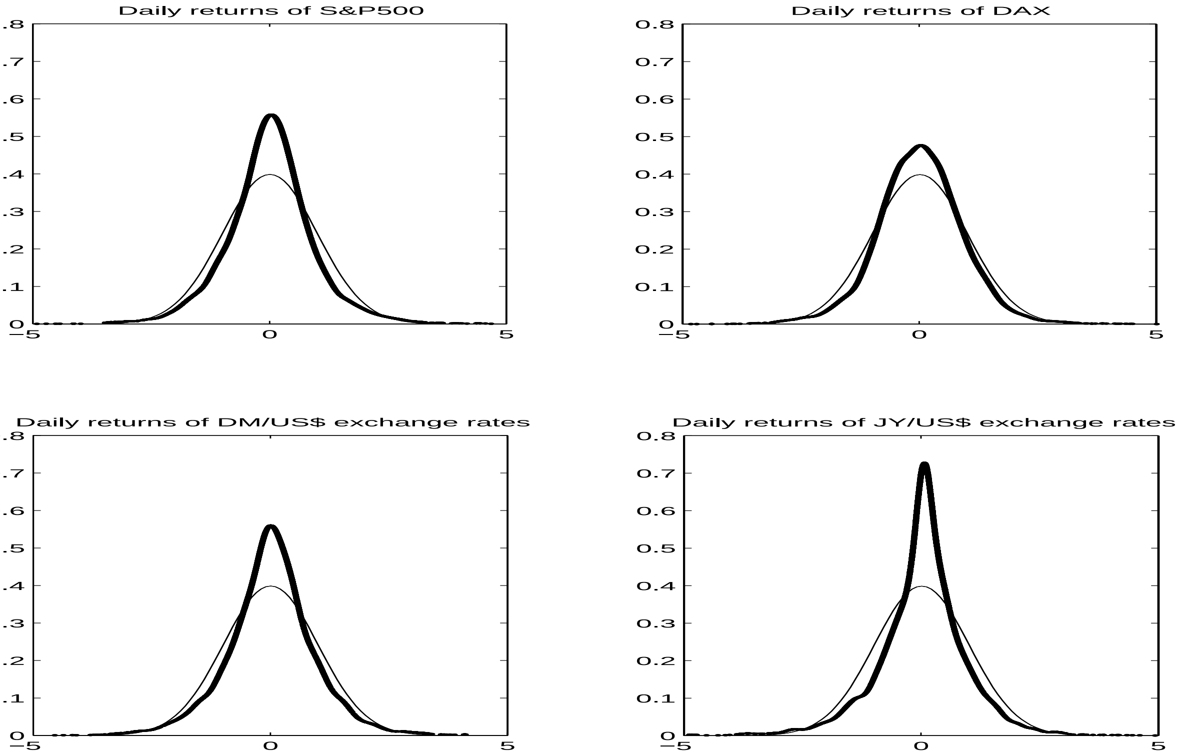

Figure 2: Empirical densities

Each of the empirical densities show excessive peaks around the mean and, at the same time, thicker tails than those of the normal density. We estimate the four times series by means of our MC-method-based test procedure. Table 2 shows the results of the estimation and the Hodge-Lehmann-type estimates and the corresponding confidence interval is graphically demonstrated in Figure 3, where the solid line gives p-values at given α of the empirical data and the three dotted lines give simulated quantiles of 90%, 95% and 99%. The estimates by means of the Hill estimation are 1.67, 1.80, 1.55 and 1.56 for S&P500, DAX, JY/US$ and DM/US$, respectively.

Table 2. Estimated tail-thickness parameter and confidence level from the MC test

Data	Estimates and corresponding confidence interval[a]
S&P500	[1.59 1.61 1.61 **1.65** 1.67 1.68 1.69]
DAX	[1.76 1.77 1.78 **1.81** 1.85 1.85 1.87]
JY/US$	[1.48 1.50 1.51 **1.54** 1.58 1.58 1.60]
DM/US$	[1.55 1.57 1.58 **1.62** 1.66 1.67 1.69]

[a]Estimates are determined as the α-value at which the p-value is the smallest (thick) and 90%-, 95%- and 99% confidence intervals.

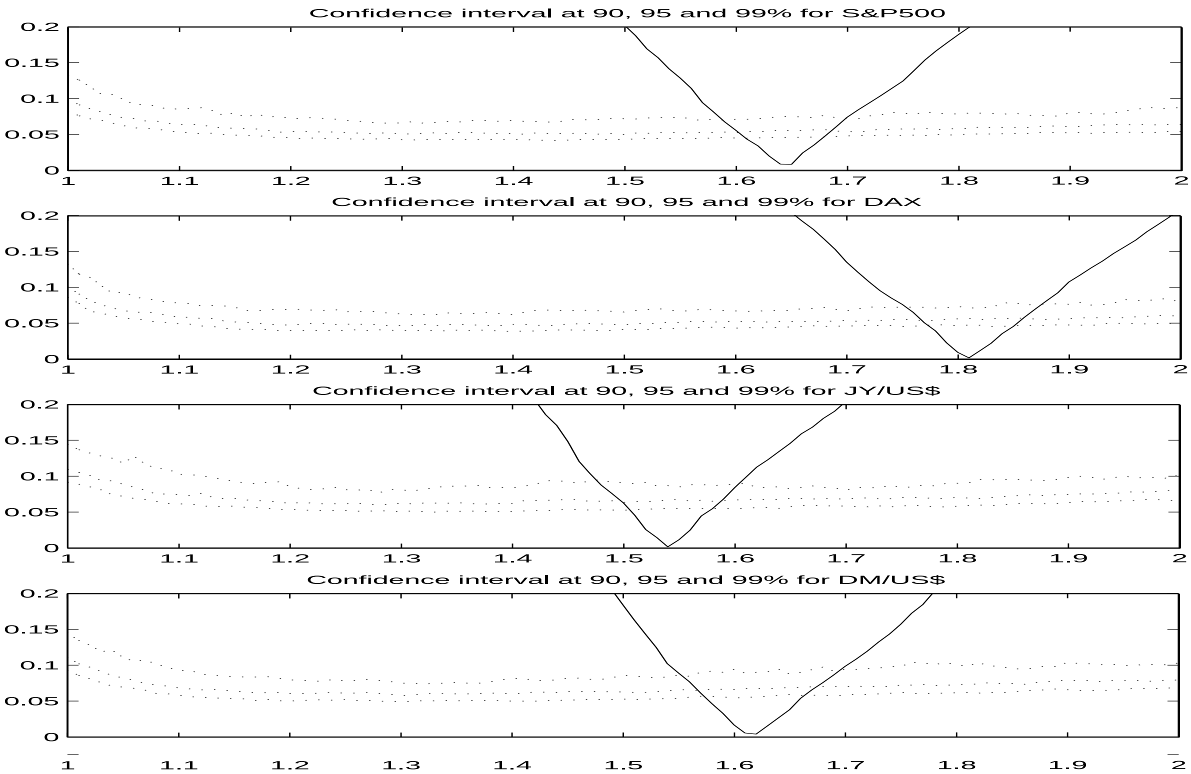

Figure 3: Hodge-Lehmann-type estimates and confidence interval

4 Concluding Remarks

In this paper, we considered estimation and test of stability parameter of α-stable distributions using an MC test. It turns out that our estimate is more efficient in the sense of mean square error than the Hill estimator even by assuming *known* k. The main result of the paper is that the MC test provides an exact confidence interval for finite samples, as demonstrated in table 2 and Figure 3. For an estimation and test for the symmetric parameter of α-stable distributions, β, based on the maximized MC method proposed by Dufour (2002) are being studied by the authors.

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