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Testing for business cycle asymmetries based on autoregressions with a Markov-switching intercept

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Abstract:

In this paper, we investigate the implications of the two concepts of asymmetry defined by Sichel (1993) - deepness and steepness - for first-order autoregressive processes with a Markov-switching intercept. In order to do so, we derive the two required formulas determining the coefficient of skewness of first-order autoregressive processes with a Markov-switching intercept and the coefficient of skewness of the first differences of these processes. For the special case of two states, we present the parameter restrictions leading to non-deepness and non-steepness. We show that these restrictions imply that the conclusions of Clements & Krolzig (2003) with respect to asymmetries of processes with a Markov-switching intercept are not correct. Finally, we apply the results to U.S. GDP which is found to exhibit strongly significant deepness and steepness.

Keywords: asymmetry, deepness, steepness, Markov-switching, business cycles

JEL-Classification: C12, C22

Non Technical Summary

The notion that business cycles exhibit asymmetries has a long history in economics. The existence of different types of asymmetric business cycle behavior has been conjectured, inter alia that recessions tend to be more pronounced and of a shorter duration than expansions, and that recoveries appear to take a more moderate course than contractions. If such asymmetries exist, their presence has important consequences for the setup of business cycle models and for our understanding of the effects of economic policy.

Most models developed for the analysis of business cycle phenomena are based on the assumption that business cycles are generated by shocks hitting a propagation mechanism. According to this concept, the economy acts as the propagation mechanism, causing single shocks to affect macroeconomic variables over a longer period. The bulk of business cycle models relies on a linear (or linearized) specification and symmetric shocks. This setup has a direct impact on the symmetry properties of business cycles generated by these models. Due to the linear specification, all variables of such models inherit the symmetry property of the shocks. Therefore, the existence of pronounced asymmetries would cast doubt on the appropriateness of this class of business cycle models. In this paper, we develop a parametric test for asymmetry.

Addressing the question of asymmetry empirically is not a trivial task. Most tests for asymmetries belong to the class of non-parametric tests. However, unless the samples of the variables under study are large, these tests often fail to detect asymmetries. In contrast to that, parametric tests typically work well also in smaller samples, but require the more restrictive assumption of a specific data generating process. Recently, Clements & Krolzig (2003) have proposed parametric tests for asymmetries based on Markov-switching processes which are widely regarded as suitable for the investigation of variables subject to business cycle fluctuations. In this work, we derive the necessary formulas for parametric tests based on a specific form of Markov-switching processes. We thereby find that certain results of Clements & Krolzig (2003) are not correct, so that wrong conclusions concerning the existence and the type of asymmetries can emerge.

Finally, we apply the tests for asymmetries to HP-filtered U.S. real gross domestic product. This variable is a good candidate for our test, since it decreases during recessions and increases during expansions. While non-parametric tests have usually failed to detect asymmetries for this variable, our test clearly indicates the presence of asymmetries. This result implies that either the shocks hitting the U.S. economy have been asymmetric or that linear business cycle models miss important features of the U.S. economy. These features could, for example, be given by capacity constraints, credit constraints or downward rigid wages.

Nicht technische Zusammenfassung

Die Vorstellung, dass Konjunkturzyklen Asymmetrien aufweisen, hat eine lange Tradition in den Wirtschaftswissenschaften. Es wurde die Existenz verschiedener Arten von Asymmetrien vermutet, unter anderem dass Rezessionen ausgeprägter und kürzer als Expansionen ausfallen und dass Aufschwünge einen moderateren Verlauf nehmen als Abschwünge. Falls solche Asymmetrien vorliegen, hat dies bedeutende Auswirkungen auf den Aufbau von Konjunkturmodellen und auch auf unser Verständnis der Wirkungsweise von Wirtschaftspolitik.

Die meisten für die Analyse von Konjunkturphänomenen entwickelten Modelle basieren auf der Annahme, dass Konjunkturzyklen durch Schocks ausgelöst werden, die auf einen Mechanismus treffen, der zu einer Fortpflanzung der Schocks führt. Dieser Mechanismus ist durch die Ökonomie selbst gegeben, deren dynamische Struktur dafür sorgt, dass ein einmaliger Schock über einen längeren Zeitraum wirkt. Ein Großteil der Konjunkturmodelle beruht auf einer linearen (oder linearisierten) Spezifikation und symmetrischen Schocks. Dieser Aufbau hat direkte Konsequenzen für die Symmetrie-Eigenschaften der Konjunkturzyklen, die von diesen Modellen erzeugt werden. Aufgrund der linearen Spezifikation dieser Modelle überträgt sich die Symmetrie-Eigenschaft der Schocks direkt auf alle Variablen der Modelle. Daher würde das Vorliegen bedeutsamer Asymmetrien die Angemessenheit dieses Modellaufbaus in Frage stellen. Im vorliegenden Papier wird ein parametrischer Test auf Asymmetrie entwickelt.

Die empirische Überprüfung des Vorliegens von Asymmetrien ist eine anspruchsvolle Aufgabe. Die meisten Tests auf Asymmetrie gehören zur Gruppe der nicht-parametrischen Tests. Diese Tests haben jedoch oft Schwierigkeiten, Asymmetrien zu erkennen, wenn die zur Verfügung stehenden Zeitreihen nicht lang genug sind. Im Gegensatz dazu funktionieren parametrische Tests üblicherweise auch für kürzere Zeitreihen gut, aber sie erfordern die restriktivere Annahme einer konkreten Form des datengenerierenden Prozesses. Kürzlich haben Clements & Krolzig (2003) parametrische Tests auf Asymmetrien vorgeschlagen, die auf Markov-Switching-Prozessen beruhen, welche für die Untersuchung von Variablen, die konjunkturelle Schwankungen durchlaufen, weithin als geeignet angesehen werden. In dieser Arbeit leiten wir die Formeln her, die im Falle einer bestimmten Form dieser Markov-Switching-Prozesse für die Tests benötigt werden. Dabei erweist sich, dass einige Ergebnisse von Clements & Krolzig (2003) nicht korrekt sind, was zu falschen Schlüssen hinsichtlich der Existenz und der Art der Asymmetrie führen kann.

Schließlich wenden wir die Tests auf Asymmetrie auf das HP-gefilterte US-amerikanische reale Bruttoinlandsprodukt an. Diese Variable bietet sich für unseren Test an, da sie im Laufe einer Rezession sinkt und im Laufe einer Expansion steigt. Während bei dieser Variable mit nicht-parametrischen Tests im Regelfall keine Asymmetrien nachgewiesen werden konnten, zeigt sich bei unserem Test, dass diese Variable eindeutig Asymmetrien aufweist. Aus diesem Ergebnis ergibt sich die Schlussfolgerung, dass die US-amerikanische Ökonomie von asymmetrischen Schocks getroffen wurde oder dass lineare Konjunkturmodelle wichtige Eigenschaften der US-amerikanischen Ökonomie nicht berücksichtigen. Diese Eigenschaften könnten beispielsweise durch Kapazitätsbeschränkungen, Kreditrestriktionen oder nach unten starre Löhne begründet sein.

Contents

1 Introduction

The question whether macroeconomic variables exhibit asymmetries over the business cycle has a long history in macroeconomic research and at least dates back to Mitchell (1927, p. 290) who stated that “Business contractions appear to be a briefer and more violent process than business expansions”. The question might be considered interesting in itself, but the answer also has important consequences for the assessment of business cycle models. If asymmetries matter, linear models with symmetric shocks do not appear to be appropriate tools for the investigation of business cycles.

Since the coefficient of skewness is a widely-used measure for the degree of asymmetry, many studies have focussed on the estimation of this coefficient in order to assess the probability that a time series is significantly asymmetric. However, as pointed out by Bai & Ng (2002), non-parametric tests for skewness tend to suffer from low power in the presence of serial correlation. Consequently, the fact that non-parametric tests fail to reject the null of zero skewness for a large set of macroeconomic variables as, for instance, documented in Bai & Ng (2002) and Psaradakis & Sola (2003) has to be regarded with caution.

A possible way to address the problem of low power is the use of parametric tests whose application, of course, requires stronger assumptions. Since Markov-switching models are considered to be adequate tools for the estimation of variables that undergo fluctuations associated with business cycles, it seems natural to investigate the implications of the parameter values of these models for asymmetries. Yet, although Markov-switching models have become increasingly popular since the seminal work of Hamilton (1989), only recently Clements & Krolzig (2003) (henceforth CK) have shown how different concepts of asymmetries are related to Markov-switching models. More precisely, in their important paper, CK derive the implications of deepness and steepness as defined by Sichel (1993) and of sharpness as defined by McQueen & Thorley (1993) for the parameters of models with a Markov-switching mean (henceforth MSM-models). Moreover, CK claim that their test for deepness is also valid for models with a Markov-switching intercept (henceforth MSI-models). However, noting that a problem appears if the linear autoregressive process has roots close to the unit circle, they state that in this case testing for deepness in MSI-models might actually yield results for steepness.

In this paper, we show that the reason for this problem is given by the fact that tests for deepness and steepness in MSI-models are *not* equivalent to tests for deepness and steepness in MSM-models. In order to do so, we will proceed as follows. In Section 2, we present the concepts of deepness and steepness, and in Section 3 we introduce the notation used for Markov-switching models. In Section 4, we investigate the relation of deepness and steepness with Markov-switching processes. We start by briefly summarizing the results of CK for MSM processes. Then the formulas for the second and third moments of first-order autoregressive MSI-processes with an arbitrary number of states are derived. These formulas are applied to an MSI-process with two states. We investigate the implications of non-deepness and non-steepness for the parameter values of such a process and compare them to the results of CK for a corresponding MSM-process. In order to illustrate that CK's conclusion concerning the deepness of MSI-processes is not correct, we show that even for the simple MSI-process considered, non-deepness can arise with parameter values that lead to deepness of MSM-processes. In Section 5, we apply the tests for deepness and steepness based on the moments derived in Section 4 to U.S. GDP. Section 6 concludes.

2 Concepts of Asymmetries

Consider a strictly stationary univariate stochastic process $\{Z_t\}$ with mean μ_Z and standard deviation σ_Z. A straightforward way to think of asymmetry is given by the possible distributional asymmetry of Z_t. According to this concept, the stationary process $\{Z_t\}$ is said to be unconditionally symmetric about the mean or by convention shortly symmetric, if the condition on the marginal distribution of Z_t

$$\Pr\left(Z_t < \mu_Z - \epsilon\right) = \Pr\left(Z_t > \mu_Z + \epsilon\right) \tag{1}$$

holds for all $\epsilon \in \mathbb{R}$. Otherwise, the process is said to be asymmetric. In order to measure the degree of asymmetry, special interest is placed on the coefficient of skewness of Z_t which is the standardized third central moment and hence defined as

$$\tau_Z = \frac{E\left[(Z_t - \mu_Z)^3\right]}{\sigma_Z^3}. \tag{2}$$

Following the terminology of Sichel (1993), the type of asymmetry that prevails if $\tau_Z < 0$ is called *deepness*, while the type of asymmetry that prevails if $\tau_Z > 0$ is called *tallness.* Hence, deep distributions are skewed to the left, whereas tall distributions are skewed to the right. If $\tau_Z = 0$ holds, the distribution is said to exhibit *non-deepness* or to be not skewed. If $\tau_Z \neq 0$ holds, but the sign of τ_Z is not of interest, we will simply speak of *deepness* of Z_t. It will always be clear from the context whether deepness refers to $\tau_Z \neq 0$ or to $\tau_Z < 0$. It should be emphasized that non-deepness is a necessary, but not a sufficient condition for symmetry of Z_t.

A related concept of asymmetry deals with the change of Z_t over time, so that the variable of interest is not Z_t itself, but rather its first-order difference $Z_t - Z_{t-1}$ which will henceforth be denoted ΔZ_t. Note that these first-order differences are not used to render the stochastic process stationary, since $\{Z_t\}$ is stationary by assumption. The stationarity of $\{Z_t\}$ implies $\mu_{\Delta Z} = 0$, where $\mu_{\Delta Z}$ denotes the mean of ΔZ_t. If ΔZ_t is symmetric we have that

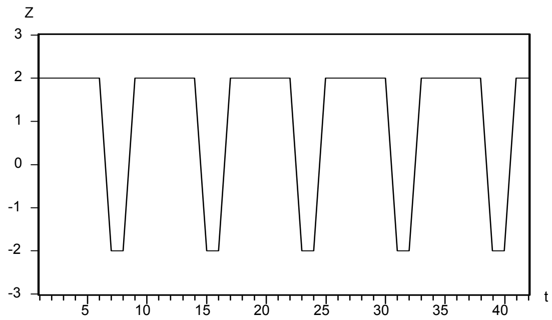
$$\Pr\left(\Delta Z_t < -\epsilon\right) = \Pr\left(\Delta Z_t > \epsilon\right) \qquad \text{for } \epsilon \in \mathbb{R} \tag{3}$$

and the degree of asymmetry of ΔZ_t can again be measured by its coefficient of skewness

$$\tau_{\Delta Z} = \frac{E\left[(\Delta Z_t)^3\right]}{\sigma_{\Delta Z}^3}. \tag{4}$$

If $\tau_{\Delta Z} = 0$ holds, Z_t is said to be non-steep. Otherwise the type of asymmetry present in Z_t is called *negative steepness* if $\tau_{\Delta Z} < 0$ and *positive steepness* if $\tau_{\Delta Z} > 0$. Again, the terminology is mainly due to Sichel (1993). Exactly as in the case of non-deepness, non-steepness of Z_t is a necessary, but not a sufficient condition for symmetry of ΔZ_t.

It is also worth noting that neither does deepness imply or prevent steepness nor does steepness imply or prevent deepness. These two concepts of asymmetry are mutually independent. In order to clarify the presented concepts of asymmetry, we present examples of a deep but non-steep process and a negatively steep but non-deep process in Figure 1.



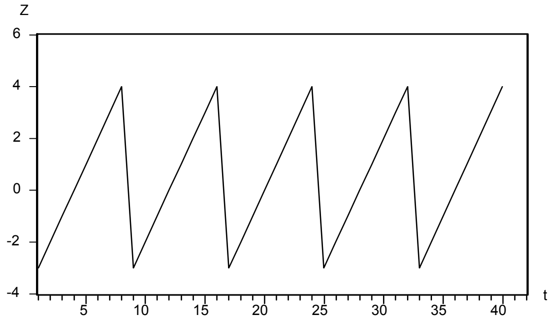


Figure 1: Deep non-steep process (left panel), and non-deep negatively steep process (right panel)

3 Markov-Switching Processes

Markov Chains Consider a univariate stochastic process $\{s_t\}$, where its state variables s_t adopt integer values i with $i \in \{1, 2, ..., m\}$. Suppose further that for this process it holds that $\Pr(s_{t+1} = i \,|s_t = j, s_{t-1} = k, ...)$ equals $\Pr(s_{t+1} = i \,|s_t = j)$ for all t, i and j. Then, the process $\{s_t\}$ is said to be an m-state first-order Markov chain, where for fixed t the variable s_t describes the state of the process in period t.[1] The probability that state s_t is succeeded by state s_{t+1} is called transition probability. All m^2 transition probabilities $p_{ji} := \Pr(s_{t+1} = i \,|s_t = j)$ are collected in an $(m \times m)$ transition matrix $\mathbf{P}$ defined by

[1]Since in what follows only first-order Markov chains will be considered, we will henceforth not mention the order unless it is necessary to avoid misunderstandings.

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{m1} \\ p_{12} & p_{22} & \cdots & p_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ p_{1m} & p_{2m} & \cdots & p_{mm} \end{bmatrix}$$

which is required to be irreducible and to have exactly one eigenvalue on the unit circle. It is useful to define a random vector $\boldsymbol{\xi}_t$ as an $(m \times 1)$ vector whose ith element is equal to unity if $s_t = i$ and whose other elements all equal zero, so that the conditional expectation of $\boldsymbol{\xi}_{t+q}$ given $\boldsymbol{\xi}_t$ can be expressed as

$$E\left[\boldsymbol{\xi}_{t+q} | \boldsymbol{\xi}_t\right] = \mathbf{P}^q \boldsymbol{\xi}_t. \tag{5}$$

Let $\mathbf{1}_m$ denote an $(m \times 1)$ vector of ones and normalize $\boldsymbol{\gamma}$ so that $\mathbf{1}_m' \boldsymbol{\gamma} = 1$ holds. Then it can be shown (see, e.g. Hamilton (1994, ch. 22)) that $\boldsymbol{\gamma}$ contains the unconditional probabilities of each state. Stated formally, this means that

$$E\left[\boldsymbol{\xi}_t\right] = \boldsymbol{\gamma}$$

holds.

Markov-Switching Processes Since the observations of most macroeconomic time series are not realizations of discrete-valued random variables, Markov chains cannot be applied directly to these observations. Instead, one must augment the Markov chain with a continuous-valued random variable, where one commonly assumes that the Markov chain governs the state variable of an otherwise standard linear stochastic process with Gaussian white noise.

Equation (5) implies that the state equation can be expressed as

$$\boldsymbol{\xi}_{t+1} = \mathbf{P} \boldsymbol{\xi}_t + \mathbf{u}_{t+1} \tag{6}$$

where the error vector $\mathbf{u}_{t+1}$ is defined by

$$\mathbf{u}_{t+1} = \boldsymbol{\xi}_{t+1} - E\left[\boldsymbol{\xi}_{t+1} \left| \boldsymbol{\xi}_t\right.\right].$$

Given the state equation, a linear measurement equation can be specified by

$$(1-\theta L) Z_t - (1-\phi L) \mu_{s_t} = \Upsilon(L) \varepsilon_t \qquad \varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right) \tag{7}$$

where $\Upsilon(L)$ is a lag polynomial given by $\Upsilon(L) = \sum_{i=0}^{q} (\upsilon_i L^i)$, L is the lag operator and θ and ϕ are scalars.[2] Furthermore, the processes of the states $\{s_t\}$ and the normally distributed error terms $\{\varepsilon_t\}$ are assumed to be independent. The term μ_{s_t} denotes the value of μ in regime s_t. For instance, if in period t state two occurs, μ_{s_t} adopts the value μ_2 in period t. For the following calculations, it is useful to define the vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \ldots \mu_m)'$, so that

$$\mu_{s_t} = \boldsymbol{\xi}_t' \boldsymbol{\mu}$$

holds. The process of Z_t described by (7) is required to be stable and will be referred to as a Markov-switching process.

If $\theta = \phi$ holds, the Markov-switching process contains a Markov-switching mean as in the well-known model set up by Hamilton (1989). We will refer to this specification as MSM-process. A different process evolves when ϕ is set to zero. This specification which can be found, for instance, in Clements & Krolzig (1998) will be referred to as MSI-process. Obviously, MSI-processes and MSM-processes would be identical in the special case of $\theta = \phi = 0$. Therefore, in order to avoid misunderstandings, we require that for MSI-processes $\theta \neq 0$ holds.

In order to give an intuition of how MSM- and MSI-processes behave, we plot an MSM-process with $\theta = \phi = 0.9$ and an MSI-process with $\theta = 0.9$ in Figure 2. For both

[2] We restrict the presentation to first-order autoregressive processes, since only for these processes the formulas for their third-order moments are derived.

processes, the number of states equals two, σ_{ε}^2 is set equal to zero, and both processes follow the same state equation. Evidently, the MSI-process slowly approaches a certain value after a state change, whereas the MSM-process immediately jumps to its new level.[3] Moreover, the value of ϕ does not matter for the dynamics of the MSM-process if σ_{ε}^2 equals zero, whereas the value of θ determines the speed of approaching a new value of Z_t for the MSI-process. A large value of θ implies slow approaching.

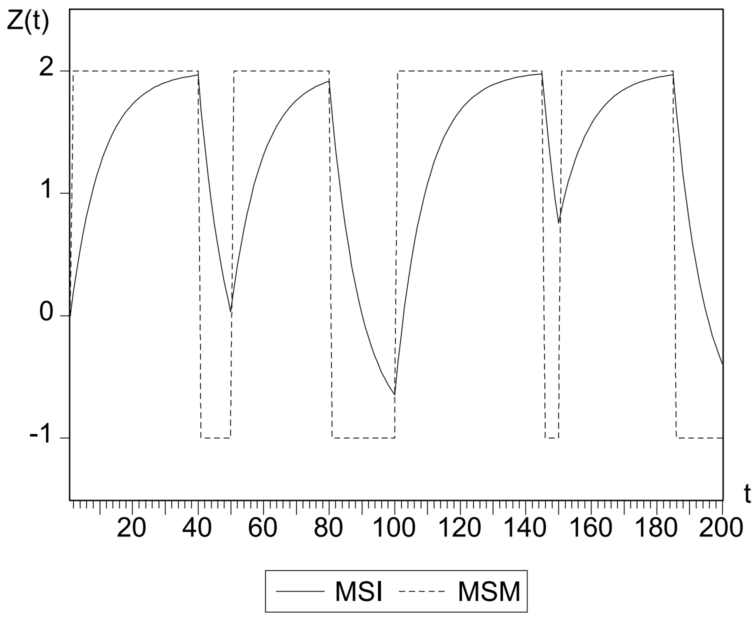


Figure 2: An MSI- and an MSM-process

[3]In this example, the processes are constructed such that the MSI-process would attain the same value as the MSM-process if no regime change occurs and t goes to infinity.

4 Relation of Asymmetries and Markov-Switching Processes

4.1 Introductory Remarks

Before we proceed, it is useful to investigate the properties of the convolution of two random variables under certain conditions. Consider two independent stationary random variables Z_t and ϵ_t, where ϵ_t has zero mean, is symmetric and hence disposes of odd central moments equal to zero.[4] Then the variance and the odd central moments of the sum of Z_t and ϵ_t are given by

$$E\left[(Z_t + \epsilon_t - \mu_Z)^2\right] = E\left[(Z_t - \mu_Z)^2\right] + E\left[\epsilon_t^2\right]$$

and

$$E\left[(Z_t + \epsilon_t - \mu_Z)^k\right] = E\left[(Z_t - \mu_Z)^k\right] \qquad \text{for } k = 1, 3, 5, \dots, \tag{8}$$

respectively. From these results, it follows that ϵ_t has no influence on the symmetry and the sign of skewness of Z_t, but only on the magnitude of the coefficient of skewness due to its influence on the variance of Z_t. Note that the normal random variables ε_t and $\Delta\varepsilon_t$, which are part of the Markov-switching process (7), as well as any linear transformation of ε_t and $\Delta\varepsilon_t$ have zero mean and are symmetric. Thus, when symmetry and the signs of deepness and steepness of Markov-switching processes are investigated, added normal error terms can be ignored during the analysis.

In order to analyze the properties with respect to symmetry and deepness of Z_t when Z_t is determined by (7), it is useful to rewrite (7) as

$$Z_t = \frac{1 - \phi L}{1 - \theta L}\mu_{s_t} + \frac{1}{1 - \theta L}\Upsilon(L)\,\varepsilon_t \tag{9}$$

[4] We assume that all odd moments exist.

and thus as the sum of two independent processes. Because of the mentioned zero mean and zero skewness of ε_t and its independence of the Markov chain, only the term $\frac{1-\phi L}{1-\theta L}\mu_{s_t}$ can cause asymmetry of Z_t. Equivalently, for ΔZ_t one obtains

$$\Delta Z_t = \frac{1-\phi L}{1-\theta L}\Delta\mu_{s_t} + \frac{1}{1-\theta L}\Upsilon\left(L\right)\Delta\varepsilon_t \tag{10}$$

where in analogy to (9) because of the symmetry, zero mean and zero skewness of $\Delta\varepsilon_t$ and its independence of the Markov chain, only the term $\frac{1-\phi L}{1-\theta L}\Delta\mu_{s_t}$ can cause asymmetry of ΔZ_t.

4.2 MSM-processes

4.2.1 An Arbitrary Number of States

The study of MSM-processes is especially easy, since for these processes the term $\frac{1-\phi L}{1-\theta L}$ appearing in (9) and (10) simply equals one. CK show that for MSM-processes with an arbitrary number of states, the condition for non-deepness can be stated as

$$E\left[\left(Z_t-\mu_Z\right)^3\right] = \boldsymbol{\gamma}'\left(\boldsymbol{\mu}-\left(\boldsymbol{\gamma}'\boldsymbol{\mu}\right)\mathbf{1}_m\right)^3 = 0$$

where taking the power of a vector means taking the power of each element of that vector. Note that the conditional probabilities, i.e. the elements of the transition matrix do not enter this formula directly.[5] According to CK, in contrast to that, non-steepness requires

$$\sum_{i=0}^{m-1}\sum_{j=i+1}^{m}\left(\gamma_i p_{ij}-\gamma_j p_{ji}\right)\left(\mu_j-\mu_i\right)^3 = 0 \tag{11}$$

to hold, so that for this concept of asymmetry also the conditional probabilities matter directly.[6]

[5] Of course, the unconditional probabilities are determined by the conditional probabilities.

[6] It should be noted that both formulas are also valid for an order of the autoregressive process different from one. In this work, however, we exclusively focus on the first-order autoregressive case.

4.2.2 Two States

In the case of two states, the third central moment of an MSM-process can be expressed as

$$E\left[(Z_t - \mu_Z)^3\right] = (\mu_2 - \mu_1)^3 (2\gamma_1 - 1)(1 - \gamma_1)\gamma_1$$

so that, since in this case the unconditional probabilities are determined by

$$\gamma_1 = \frac{1 - p_{22}}{2 - p_{11} - p_{22}}$$

and

$$\gamma_2 = \frac{1 - p_{11}}{2 - p_{11} - p_{22}} = 1 - \gamma_1,$$

non-deepness of Z_t occurs when the transition matrix of the states is symmetric, i.e. if $p_{11} = p_{22}$ holds. Furthermore, considering the unconditional density of an MSM-process, which can, for instance, be found in Hamilton (1994, ch. 22), it is evident that non-deepness also implies symmetry. Applying (11) to the case of two states yields

$$E\left[(\Delta Z_t - \mu_{\Delta Z})^3\right] = 0$$

which implies that MSM-processes with two states are non-steep regardless of their parameter values.

4.3 MSI-Processes

4.3.1 An Arbitrary Number of States

Concerning the skewness of MSI-processes, to the best of our knowledge no formulas are documented in the literature, presumably because the derivation of the third central moment is extremely cumbersome. In order to inspect the relation of the parameters and the deepness and steepness of MSI-processes, we hence first have to derive the formulas for the third central moments of these processes.

A formula for the second central moments of MSI-processes can be found in Krolzig (1997). Since the formula presented there, however, is based on Markov-switching processes in their unrestricted form, we also present a formula that is more convenient if the Markov-switching process is given in its restricted form.[7]

Deepness:

Proposition 1 *The second noncentral moment of an MSI-process with* $\sigma_\varepsilon^2 = 0$ *is given by*

$$E\left[Z_t^2\right] = \frac{1}{1-\theta^2}\gamma'\boldsymbol{\mu}^2 + \frac{2}{1-\theta^2}\left(\boldsymbol{\mu}'\theta\mathbf{P}\left(\mathbf{I}-\theta\mathbf{P}\right)^{-1}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right) \tag{12}$$

where the operator $\odot$ *denotes the Schur product.*

Proof. See Appendix A.1. ■

Proposition 2 *The third noncentral moment of an MSI-process is given by*

$$E\left[Z_t^3\right] = \left(\begin{array}{c} \frac{1}{1-\theta^3}\gamma'\boldsymbol{\mu}^3 \\ +\frac{3}{1-\theta^3}\left(\boldsymbol{\mu}^2\right)'\left(\theta\mathbf{P}\left(\mathbf{I}-\theta\mathbf{P}\right)^{-1}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right) \\ +\frac{3}{1-\theta^3}\boldsymbol{\mu}'\left(\theta^2\mathbf{P}\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}^2\right) \\ +\frac{6}{1-\theta^3}\mathbf{1}_m'\left(\mathbf{A}_1\odot\left(\theta^2\mathbf{P}\right)\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\right)\left(\mathbf{A}_2\odot\mathbf{A}_1\odot\left(\theta\mathbf{P}\right)\left(\mathbf{I}-\theta\mathbf{P}\right)^{-1}\right)\mathbf{1}_m \end{array}\right), \tag{13}$$

where the matrices $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are defined by*

$$\begin{aligned} \mathbf{A}_1 &: = \boldsymbol{\mu}\otimes\mathbf{1}_m' \\ \mathbf{A}_2 &: = \left(\boldsymbol{\mu}\odot\boldsymbol{\gamma}\right)'\otimes\mathbf{1}_m \end{aligned} \tag{14}$$

[7] In the restricted form, the elements of the columns of $\mathbf{P}$ as well as the elements of $\boldsymbol{\gamma}$ sum up to one. In the unrestricted form, this restriction is eliminated. In the literature, the restricted form is encountered far more frequently than the unrestricted form.

where the operator $\otimes$ denotes the Kronecker product.

Proof. See Appendix A.1. ■

With the mean of an MSI-process given by

$$\mu_Z = \frac{1}{1-\theta}\boldsymbol{\gamma}'\boldsymbol{\mu}, \tag{15}$$

the central second and third moment of Z_t can be calculated easily by using

$$E\left[(Z_t - \mu_Z)^2\right] = E\left[Z_t^2\right] - \mu_Z^2 + \sigma^2_{\frac{1}{1-\theta L}\Upsilon(L)\varepsilon}, \tag{16}$$

where $\sigma^2_{\frac{1}{1-\theta L}\Upsilon(L)\varepsilon}$ denotes the variance of $\frac{1}{1-\theta L}\Upsilon(L)\varepsilon_t$, and

$$E\left[(Z_t - \mu_Z)^3\right] = E\left[Z_t^3\right] - 3E\left[Z_t^2\right]\mu_Z + 2\mu_Z^3, \tag{17}$$

which in turn allow the determination of the deepness of Z_t.

Steepness: To check for non-steepness of a two-state MSI-process, one needs to consider the equation

$$\Delta Z_t = (1-\theta L)^{-1}\Delta\mu_{s_t} = \sum_{i=0}^{\infty}(\theta L)^i\Delta\mu_{s_t} \tag{18}$$

or equivalently

$$\Delta Z_t = \mu_{s_t} + (\theta - 1)\sum_{i=0}^{\infty}(\theta L)^i\mu_{s_{t-1}}. \tag{19}$$

While with (19) one can operate directly on the variables related to Z_t in order to study the coefficient of skewness of ΔZ_t, an investigation of (18) requires the definition of new states related to ΔZ_t in order to obtain a first-order Markov chain. Nevertheless, operating on (18) can turn out to be a more elegant solution, since all that is needed for the investigation of the sign of steepness is the redefinition mentioned and the insertion of the obtained intercept vector $\boldsymbol{\mu}_{\Delta}$ and transition matrix $\mathbf{P}_{\Delta}$ into (13). Since the expectation of ΔZ_t is zero, this procedure directly yields the third central moment of ΔZ_t. If one is interested in the coefficient of skewness of ΔZ_t, one calculates the variance by inserting $\boldsymbol{\mu}_{\Delta}$ and $\mathbf{P}_{\Delta}$ into (12), where one in addition has to add the variance of $\frac{1}{1-\theta L}\Upsilon(L)\Delta\varepsilon_t$ which will henceforth be denoted $\sigma^2_{\frac{1}{1-\theta L}\Upsilon(L)\Delta\varepsilon}$.

Equivalent formulas for the second and third moment of ΔZ_t based on the states of the process of Z_t, i.e. on the intercept vector $\boldsymbol{\mu}$ and the transition matrix $\mathbf{P}$, are presented below.

Proposition 3 *If Z_t is described by an MSI-process, the second central moment of ΔZ_t is given by*

$$E\left[(\Delta Z_t)^2\right] = \boldsymbol{\gamma}'\boldsymbol{\mu}^2 + 2(\theta-1)\boldsymbol{\mu}'(\mathbf{I}-\theta\mathbf{P})^{-1}\mathbf{P}(\boldsymbol{\gamma}\odot\boldsymbol{\mu}) + (\theta-1)^2 E\left[Z_t^2\right] + \sigma^2_{\frac{1}{1-\theta L}\Upsilon(L)\Delta\varepsilon} \tag{20}$$

Proof. See Appendix A.2. ■

Proposition 4 *If Z_t is described by an MSI-process, the third central moment of ΔZ_t is given by*

$$E\left[(\Delta Z_t)^3\right]$$
$$= E\left[\begin{array}{c} \boldsymbol{\gamma}'\boldsymbol{\mu}^3 \\ +3(\theta-1)(\boldsymbol{\mu}^2)'(\mathbf{I}-\theta\mathbf{P})^{-1}\mathbf{P}(\boldsymbol{\gamma}\odot\boldsymbol{\mu}) \\ +3(\theta-1)^2\left[\begin{array}{c} \boldsymbol{\mu}'\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\mathbf{P}(\boldsymbol{\gamma}\odot\boldsymbol{\mu}^2) \\ +2\cdot\mathbf{1}_m'\left(\mathbf{A}_1\odot\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\mathbf{P}\right)\left(\mathbf{A}_2\odot\mathbf{A}_1\odot(\theta\mathbf{P})(\mathbf{I}-\theta\mathbf{P})^{-1}\right)\mathbf{1}_m \end{array}\right] \\ +(\theta-1)^3 E\left[Z_t^3\right] \end{array}\right], \tag{21}$$

where again, $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined by (14).

Proof. See Appendix A.2. ■

Evidently, the coefficient of skewness of ΔZ_t can be calculated easily from (20) and (21). The application of these formulas might be more convenient if the definition of $\boldsymbol{\mu}_\Delta$ and $\mathbf{P}_\Delta$ appears difficult.

4.3.2 Two States

According to CK, testing for deepness of MSI-processes is equivalent to testing for deepness of MSM-processes, unless the roots of the autoregressive process are close to the unit circle. In the latter case, CK state that using the restrictions for non-deepness of MSM-processes implies non-steepness of MSI-processes instead of non-deepness. This means that for two-state MSI-processes, $p_{11} = p_{22}$ would be the only restriction for non-deepness if θ is not too large in absolute value. However, if θ is large in absolute value, $p_{11} = p_{22}$ would be the restriction for non-steepness.

In what follows, we will show that MSI-processes can be non-deep if $p_{11} \neq p_{22}$. Moreover,we will show that $p_{11} = p_{22}$ is the restriction for non-steepness for all $\theta \neq 0$.

Deepness: For the special case of two states, application of (12) and (13) shows that the second central moment simplifies to

$$E\left[(Z_t-\mu)^2\right] = (\mu_2-\mu_1)^2 \frac{(1-p_{22})(1-p_{11})(1+\theta(p_{11}+p_{22}-1))}{\left(1-\theta^2\right)(1-\theta(p_{11}+p_{22}-1))(2-p_{11}-p_{22})^2} + \sigma^2_{\frac{1}{1-\theta L}\Upsilon(L)\varepsilon} \tag{22}$$

and the third central moment becomes

$$E\left[(Z_t-\mu)^3\right] = (\mu_2-\mu_1)^3 \frac{(1-p_{22})(1-p_{11})(p_{11}-p_{22})\left((p_{11}+p_{22}-1)^2\theta^3+2(p_{11}+p_{22}-1)\left(\theta^2+\theta\right)+1\right)}{(1-\theta)\left(\theta^2(1-p_{11}-p_{22})+1\right)(\theta(1-p_{11}-p_{22})+1)(2-p_{11}-p_{22})^3\left(\theta^2+\theta+1\right)}. \tag{23}$$

Thus, in contrast to the MSM-processes, it is not possible to determine the third central moment using only the unconditional probabilities.[8]

In order to generate non-deepness for an MSI-process with two states, (23) must equal

[8]Actually, the same holds for the second central moments which equal $E\left[(Z_t-\mu_Z)^2\right] = \boldsymbol{\gamma}'\left(\boldsymbol{\mu}-(\boldsymbol{\gamma}'\boldsymbol{\mu})\,\mathbf{1}_m\right)^2$ in the case of MSM-processes.

zero. Since we are not considering cases where either p_{11} or p_{22} equal one,[9] it follows that only the solutions

$$p_{11} = p_{22} \tag{24}$$

and

$$p_{11} = 1 - p_{22} - \frac{1}{\theta}\left(\frac{1}{\theta} + 1\right) + \frac{1}{\theta^2}\sqrt{\theta^2 + \theta + 1} \tag{25}$$

are feasible. While the non-deepness condition stated in (24) is identical for MSM-processes and identical to the condition for symmetry of two-state MSM- and MSI-processes which is given by $p_{11} = p_{22}$,[10] condition (25) has no counterpart in the case of MSM-processes, is unrelated to symmetry and therefore emerges unexpectedly. This second possibility contradicts CK's claim that testing for non-deepness of MSI-processes is directly related to testing for non-deepness of MSM-processes.

Taking into account stability restrictions of the process and the restrictions placed on $\mathbf{P}$, one obtains that for

$$\theta \in (0.382, 1)$$

condition (25) yields admissible combinations of p_{11} and p_{22}. Inserting this restriction for θ into (25), it turns out that

$$p_{11} + p_{22} \in (0, 0.732)\,. \tag{26}$$

has to be fulfilled if (25) holds.[11] Equation (25) can also be illustrated graphically as done in Figure 3. The line represents all combinations of θ and $p_{11} + p_{22}$ generating zero deepness. It is easy to show that combinations on opposite sides of that line give rise to

[9] This follows from the conditions imposed on the transition matrix.

[10] In Appendix A.3 we show that $\gamma_1 = \gamma_2$ and hence $p_{11} = p_{22}$ is the condition for symmetry for two-state MSI-processes.

[11] The precise intervals for θ and $p_{11} + p_{22}$ are given by $\theta \in \left(\frac{3}{2} - \frac{1}{2}\sqrt{5}, 1\right)$ and $p_{11} + p_{22} \in \left(0, -1 + \sqrt{3}\right)$.

opposite signs of the third central moment when the signs of $\mu_2 - \mu_1$ and $p_{11} - p_{22}$ are identical for these combinations. Thus, the sign of the third central moment depends on the sign of the expression

$$(p_{11} + p_{22} - 1)^2 \theta^3 + 2(p_{11} + p_{22} - 1)\left(\theta^2 + \theta\right) + 1. \qquad (27)$$

Suppose that μ_1 and p_{11} are larger than their respective counterpart of state 2. Then formula (23) implies that, if expression (27) is positive, the third central moment is negative and vice versa. The values that expression (27) can take multiplied by -1 are depicted in Figure 4. In the left panel, only values smaller than zero which thus cause negative third central moments are plotted, whereas in the right panel only positive values are displayed. Note that for most combinations of θ and $p_{11}+p_{22}$, the third central moment adopts negative values under the mentioned conditions on the signs of $\mu_1 - \mu_2$ and $p_{11} - p_{22}$. Nevertheless, in contrast to MSM-processes, it is not sufficient to know the signs of $\mu_1 - \mu_2$ and $p_{11} - p_{22}$ in order to determine the sign of the coefficient of skewness of an MSI-process.

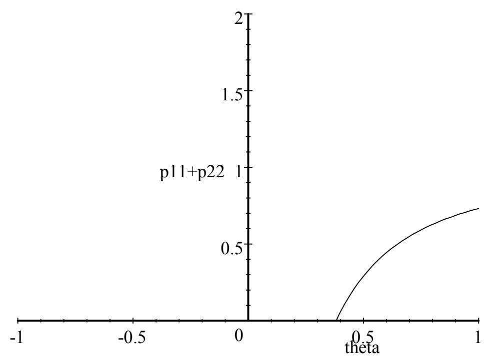


Figure 3: A condition for non-deepness in MSI(1)-models

For the purpose of hypotheses tests dealing with macroeconomic data, it is evidently very important to know that non-deepness of MSI-processes with two states can actually arise without a symmetric transition matrix. Suppose that a likelihood-ratio test for

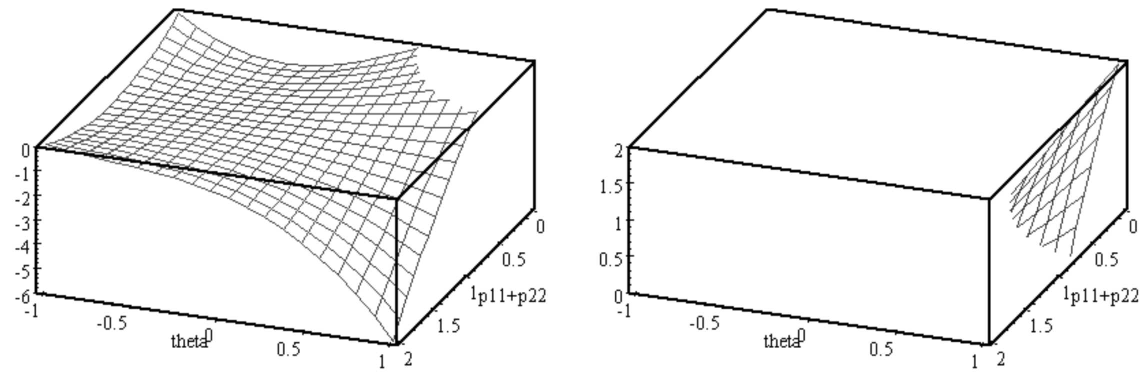


Figure 4: Expression determining the sign of $E\left[(Z_t-\mu)^3\right]$

non-deepness is to be performed. Then, *two* restricted models have to be estimated, one restricted by (24) and the other one restricted by (25). Only the model with the larger value of the log-likelihood function then has to be tested against the unrestricted model. Considering only restriction (24) as proposed by CK can therefore lead to wrong conclusions.

Finally, note that the skewness of a two-state MSI-process without Gaussian white noise can be determined by means of (22) and (23) and is hence given by

$$\tau_Z = sign\left(\mu_2-\mu_1\right)\frac{(1+\theta)\sqrt{\left(1-\theta^2\right)}\sqrt{1-\theta(p_{11}+p_{22}-1)}(p_{11}-p_{22})\left((p_{11}+p_{22}-1)^2\theta^3+2(p_{22}+p_{11}-1)\left(\theta^2+\theta\right)+1\right)}{\sqrt{(1-p_{22})}\sqrt{(1-p_{11})}\left(\sqrt{1+\theta(p_{11}+p_{22}-1)}\right)^3\left(\theta^2+\theta+1\right)\left(1-(p_{11}+p_{22}-1)\theta^2\right)} \tag{28}$$

where $sign\left(z\right)$ is defined by

$$sign\left(z\right)=\begin{cases}\dfrac{z}{|z|}\text{ if } z\neq 0\\ 0 \text{ if } z=0\end{cases}.$$

Steepness: As mentioned above, there are two possibilities to calculate the moments of ΔZ_t. If one is interested in a first-order Markov process for $\Delta\mu_{s_t}$, one needs to define four new states

$$\begin{aligned}
\{\tilde{s}_t = 1\} &: = \{s_t = 1\} \cap \{s_{t-1} = 2\} \\
\{\tilde{s}_t = 2\} &: = \{s_t = 1\} \cap \{s_{t-1} = 1\} \\
\{\tilde{s}_t = 3\} &: = \{s_t = 2\} \cap \{s_{t-1} = 2\} \\
\{\tilde{s}_t = 4\} &: = \{s_t = 2\} \cap \{s_{t-1} = 1\}
\end{aligned}$$

with associated mean vector $\boldsymbol{\mu}_\Delta$ and transition matrix $\mathbf{P}_\Delta$ given by

$$\boldsymbol{\mu}_\Delta := \begin{bmatrix} \mu_1 - \mu_2 \\ 0 \\ 0 \\ \mu_2 - \mu_1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_\Delta = \begin{bmatrix} 0 & 0 & 1 - p_{22} & 1 - p_{22} \\ p_{11} & p_{11} & 0 & 0 \\ 0 & 0 & p_{22} & p_{22} \\ 1 - p_{11} & 1 - p_{11} & 0 & 0 \end{bmatrix}. \tag{29}$$

The formulas presented for deepness can now be applied to the process for ΔZ_t described by (18) in order to derive the moments of the first-order differences of two-state MSI-processes. If one does not want to define new states, one can use the formulas (20) and (21) for the same purpose.

Using either of the proposed methods yields

$$E\left[(\Delta Z_t)^2\right] = (\mu_2 - \mu_1)^2 \tfrac{2(1-p_{11})(1-p_{22})}{(2-p_{11}-p_{22})(1-(p_{22}+p_{11}-1)\theta)(\theta+1)} + \sigma^2_{\frac{1}{1-\theta L}\Upsilon(L)\Delta\varepsilon} \tag{30}$$

and

$$E\left[(\Delta Z_t)^3\right] = (\mu_2 - \mu_1)^3 \tfrac{3\theta(1-p_{11})(1-p_{22})(p_{11}-p_{22})}{\left(\theta^2+\theta+1\right)(2-p_{11}-p_{22})\prod_{i=1}^{2}\left(\theta^i(1-p_{22}-p_{11})+1\right)} . \tag{31}$$

For $p_{11}, p_{22} \in [0, 1)$ the term (31) equals zero if and only if $p_{11} = p_{22}$. Hence, in contrast to two-state MSM-processes, a two-state MSI-process can display steepness. Moreover, testing for deepness of MSM-processes is equivalent to testing for steepness of MSI-processes for *every* $\theta \neq 0$ and not only if the absolute value of θ is close to 1 or -1 as claimed by

CK.[12]

Using (30) and (31) it can be easily verified that the coefficient of steepness without Gaussian white noise is calculated as

$$\tau_{\Delta Z} = sign\left(\mu_2 - \mu_1\right) \tfrac{3}{4} \tfrac{\sqrt{2(2-p_{11}-p_{22})}}{\theta^2+\theta+1} \tfrac{\sqrt{1-(p_{22}+p_{11}-1)\theta}}{1-(p_{22}+p_{11}-1)\theta^2} \tfrac{\theta(p_{11}-p_{22})\left(\sqrt{(\theta+1)}\right)^3}{\sqrt{(1-p_{11})}\sqrt{(1-p_{22})}} .$$

If one again supposes that μ_1 is larger than μ_2 and that $\theta > 0$ holds, negative steepness occurs if and only if p_{11} is larger than p_{22}. Thus, if $p_{11} + p_{22}$ is larger than $-1 + \sqrt{3}$ and θ is larger than zero, deepness implies negative steepness, and tallness implies positive steepness. With θ again larger than zero but $p_{11} + p_{22}$ smaller than $-1 + \sqrt{3}$, the relation between deepness and steepness is ambiguous and depends on the value of θ.

5 Application

In this section we apply the results derived for the deepness and steepness of MSI-processes with two states to U.S. real GDP. In order to render U.S. GDP stationary, we employ the widely-used filter proposed by Hodrick & Prescott (1997) (henceforth HP-filter).[13] HP-filtered output series are especially well-suited for the application of MSI-processes because they decrease during contractions and increase during expansions. Different non-parametric tests for deepness of HP-filtered U.S. output with ambiguous results have been conducted before by Sichel (1993), Canova (1998) and Razzak (2001). Sichel (1993) and Canova (1998) find no evidence for deepness of postwar U.S. real GNP. Razzak (2001) splits a postwar sample of U.S. real GDP into the exchange rate regimes of Bretton Woods and free floating, and he discovers significant deepness in the latter, but not in the former regime. To the best of our knowledge, steepness of HP-filtered U.S. output has not been tested yet.

Our raw data covers the period from the first quarter of 1955 through the fourth quarter of 2002. After taking logarithms, HP-filtering and multiplying by 100, we drop

[12] To be precise, "being equivalent" means that both tests imply the identical restrcition $p_{11} = p_{22}$.

[13] We use the common value of 1600 for the smoothing parameter.

the first and last twelve observations of the filtered series as suggested by Baxter & King (1999), so that the series investigated starts from the first quarter of 1958 and ends by the fourth quarter of 1999. The resulting series of HP-filtered U.S. real GDP is displayed in the left panel of Figure 5.

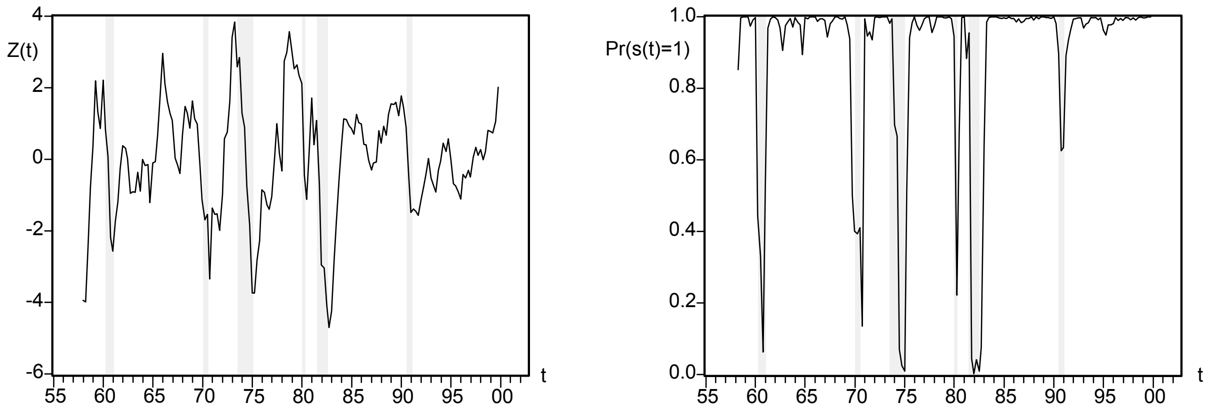


Figure 5: HP-filtered U.S. GDP (left panel) and smoothed probabilities of being in the expansionary regime (right panel). Shaded areas indicate recessions as dated by the NBER.

As emphasized by CK, the results of parametric tests depend strongly on the assumptions about the stochastic process one intends to study. Thus, the appropriateness of the specification chosen has to be investigated extremely carefully. Applying specification tests to Markov-switching models is a challenging task, since many standard tests rely on the normality of the residuals under the null hypothesis. For Markov-switching models, however, normality of the residuals is not a valid assumption as noted by Krolzig (1997, pp. 132-133). In this paper, we employ the Newey-Tauchen-White tests for dynamic misspecification proposed by Hamilton (1996). These tests tend to overreject in small samples so that we can have some confidence in the appropriateness of our specification if the tests do not reject the null of correct specification.

We choose to estimate a two state MSI-model with one moving-average parameter in order to model HP-filtered U.S. real GDP. The estimated parameters of this model and

Table 1: Estimation results for two-state MSI-models with one moving-average parameter

	unrestricted model	restriction $p_{11} = p_{22}$	restriction (25)
μ_1	0.208	0.215	0.306
	(0.076)	(0.083)	(0.603)
μ_2	-1.324	-1.201	-0.098
	(0.306)	(0.294)	(0.239)
σ_ε^2	0.462	0.473	0.613
	(0.060)	(0.067)	(0.103)
θ	0.769	0.763	0.808
	(0.048)	(0.051)	(0.049)
υ	0.128	0.123	0.234
	(0.083)	(0.087)	(0.080)
p_{11}	0.956	0.935	0.071
	(0.024)	(0.033)	(0.542)
p_{22}	0.655	0.935	0.563
	(0.145)	-	-
$\mathcal{L}$	-193.976	-199.911	-199.232

Note: Standard errors are in parentheses and computed based on the Hessian. $\mathcal{L}$ denotes the value of the log-likelihood function.

the value of the log-likelihood function are presented in the first column Table 1. The corresponding results of the Newey-Tauchen-White tests are displayed in Table 2. Evidently, there are no signs of significant misspecification in the form of residual autocorrelation, autoregressive conditional heteroskedasticity (denoted ARCH effects), violation of the first-order Markov specification or varying transition probabilities.[14] Moreover, the estimated probabilities of being in a certain state are found to match the business cycle pattern published by the National Bureau of Economic Research (henceforth NBER) very well. This can be verified simply by looking at the smoothed probabilities of being in the expansionary state and the recessionary periods as dated by the NBER which are displayed in the right panel of Figure 5.

Having found a well-specified Markov-switching model, we proceed by estimating re-

[14]The test for violation of the first-order Markov specification basically tests whether the equality $\Pr(s_t \,|s_{t-1}) = \Pr(s_t \,|s_{t-1}, s_{t-2})$ holds. Similarly, the test for varying transition probabilities tests whether the assumption of $\Pr(s_t \,|s_{t-1}) = \Pr(s_t \,|s_{t-1}, \varepsilon_{t-1})$ is correct. More details concerning the specification tests performed are available upon request.

Table 2: Newey-Tauchen-White tests for dynamic misspecification

H_0	p-value
no residual autocorrelation	0.198
no ARCH effects	0.215
appropriateness of first-order Markov specification	0.835
constant transition probabilities	0.212

Table 3: LR tests for deepness and steepness

restriction	LR-statistic	p-value
non-deepness, i.e. (25)	10.512	0.0012
non-steepness, i.e. $p_{11} = p_{22}$	11.870	0.0006

stricted MSI-models in order to perform subsequent likelihood-ratio tests (henceforth LR tests) for deepness and steepness. As shown above, the test for deepness requires the estimation of two restricted models, where one model is restricted by equality of the transition probabilities p_{11} and p_{22} and the other is restricted by equation (25). Since testing for steepness only requires an estimation with the restriction $p_{11} = p_{22}$, no additional models have to be estimated for this test. The validity of the second condition for non-steepness $\theta = 0$ will be assessed using a Wald test. The parameter values and the values of the log-likelihood functions are displayed in the second and third column of Table 1. Evidently, the value of the log-likelihood function for the model with restriction (25) is larger than for the model with restriction $p_{11} = p_{22}$, so that the test for deepness requires a restriction that is different from the restriction of the test for steepness.

As can be inferred from Table 1, Wald tests reject the null of $\theta = 0$ at a significance level of virtually zero. The results of the LR tests are displayed in Table 3. Evidently, both null hypotheses are rejected at all common significance levels, since the p-values are decisively lower than one percent in both cases. Non-steepness would even be rejected at a significance level of 0.1 percent, whereas the p-value of the test for deepness exceeds this value by a small amount.

6 Summary and Concluding Remarks

In this paper, we have derived the formulas for the coefficient of skewness of first-order autoregressive processes with a Markov-switching intercept (MSI-processes), where the underlying Markov chain has an arbitrary number of states. We have also shown how to determine the coefficient of skewness of the first-order differences of MSI-processes. For the special case of two states, we have presented the parameter restrictions for non-deepness and non-steepness.

Our results imply that there are two different combinations of parameter values leading to non-deepness of two-state MSI-processes, where only one of these combinations is identical to the restriction for non-deepness of autoregressive processes with a two-state Markov-switching mean (MSM-processes). Hence, our results show that the conclusions of Clements & Krolzig (2003) with respect to MSI-processes are not correct, since they claim that testing for deepness of MSI-processes is equivalent to testing for deepness of MSM-processes. Moreover, we find that, in contrast to two-state MSM-processes, two-state MSI-processes in general exhibit steepness, and that the restriction for non-steepness of a two-state MSI-process is equivalent to the restriction for non-deepness of a two-state MSM-process. Thus, applying the parameter restriction for non-deepness of two-state MSM-processes to two-state MSI-processes *always* results in testing for steepness, and not only if the autoregressive process has roots close to the unit circle, as claimed by Clements & Krolzig (2003).

Finally, we have applied our results to postwar U.S. real GDP which was detrended by means of the HP-filter. Specification tests indicate that a two-state MSI-model with one moving-average term provides an appropriate description of the data. Estimating two corresponding restricted models that imply non-deepness and non-steepness, we find that the null hypotheses of non-deepness and non-steepness are rejected unambiguously by likelihood-ratio tests. It should be noted that this result calls into question the appropriateness of linear business cycle models with non-deep and non-steep shocks, as e.g. normal shocks. Such models cannot replicate the asymmetries found in the data which might indicate the lack of an important feature of the propagation mechanism if shocks are assumed to be non-deep and non-steep. Possible modifications for these models in order to generate deepness are given-by non-linear mechanisms that dampen the effects of positive shocks (e.g. capacity constraints) or amplify the effects of negative shocks (e.g. credit constraints). Steepness could be caused by non-linear mechanisms affecting the changes of variables, as for example downward rigid wages which might give rise to negative steepness of labor input, thereby causing negative steepness of GDP.

An evident direction for future research is given by the extension of the formulas presented for second and third moments to processes with a Markov-switching intercept and an arbitrary order of the autoregressive process.

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A Appendix

A.1 Deepness of MSI-processes

Suppose that $Z_t = \frac{1}{1-\theta L}\mu_{s_t}$, that the process started in period 0 and that the last period is period T.

Second noncentral moment: Start by rewriting $E\left[Z_t^2\right]$ as

$$E\left[Z_t^2\right] = E\left[\left(\sum_{i=0}^{T} \theta\mu_{s_{t-i}}\right)^2\right]$$

$$= E\begin{bmatrix} \theta^0\theta^0\mu_{s_t}\mu_{s_t}+ & \theta^1\theta^0\mu_{s_{t-1}}\mu_{s_t}+ & \theta^2\theta^0\mu_{s_{t-2}}\mu_{s_t}+ & \ldots+ & \theta^T\theta^0\mu_{s_{t-T}}\mu_{s_t}+ \\ \theta^0\theta^1\mu_{s_t}\mu_{s_{t-1}}+ & \theta^1\theta^1\mu_{s_{t-1}}\mu_{s_{t-1}}+ & \theta^2\theta^1\mu_{s_{t-2}}\mu_{s_{t-1}}+ & \ldots+ & \theta^T\theta^1\mu_{s_{t-T}}\mu_{s_{t-1}}+ \\ \theta^0\theta^2\mu_{s_t}\mu_{s_{t-2}}+ & \theta^1\theta^2\mu_{s_{t-1}}\mu_{s_{t-2}}+ & \theta^2\theta^2\mu_{s_{t-2}}\mu_{s_{t-2}}+ & \ldots+ & \theta^T\theta^2\mu_{s_{t-T}}\mu_{s_{t-2}}+ \\ & & \vdots & & \\ \theta^0\theta^T\mu_{s_t}\mu_{s_{t-T}}+ & \theta^1\theta^T\mu_{s_{t-1}}\mu_{s_{t-T}}+ & \theta^2\theta^T\mu_{s_{t-2}}\mu_{s_{t-T}}+ & \ldots+ & \theta^T\theta^T\mu_{s_{t-T}}\mu_{s_{t-T}} \end{bmatrix}.$$

Assume that in period 0, the state is determined according to the unconditional probabilities, so that

$$\Pr\left(\mu_{s_{t-i}}\mu_{s_{t-(i+k)}}\right) = \Pr\left(s_{t-i}|s_{t-(i+k)}\right)\Pr\left(s_{t-(i+k)}\right)$$

holds for all integers i. Hence, we can set $i = 0$ and write

$$\Pr\left(\mu_{s_{t-i}}\mu_{s_{t-(i+k)}}\right) = \Pr\left(s_t|s_{t-k}\right)\Pr\left(s_{t-k}\right)$$

and

$$E\left[\mu_{s_{t-i}}\mu_{s_{t-(i+k)}}\right] = E\left[\mu_{s_t}\mu_{s_{t-k}}\right].$$

It follows that

$$E\left[Z_t^2\right] = \left(\sum_{i=0}^{T} \theta^{2i}\right) E\left[\mu_{s_t}\mu_{s_t}\right] + \sum_{k=1}^{T}\left(2\sum_{i=0}^{T-k}\theta^{2i+k} E\left[\mu_{s_t}\mu_{s_{t-k}}\right]\right)$$

holds. Since we have that

$$E\left[\mu_{s_t}\mu_{s_t}\right] = \boldsymbol{\gamma}'\boldsymbol{\mu}^2$$

and

$$\begin{aligned}
E\left[\mu_{s_t}\mu_{s_{t-k}}\right] &= \sum_{i}^{m}\sum_{i=j}^{m} \Pr\left(s_t = i, s_{t-k} = j\right)\mu_i\mu_j \\
&= \sum_{i}^{m}\sum_{j}^{m} \Pr\left(s_t = i\right)\Pr\left(s_{t-k} = j | s_t = i\right)\mu_i\mu_j \\
&= \gamma_1 \Pr\left(s_{t-k}{=}1|s_t{=}1\right)\mu_1\mu_1 + \gamma_2 \Pr\left(s_{t-k}{=}1|s_t{=}2\right)\mu_2\mu_1 + ... + \gamma_m \Pr\left(s_{t-k}{=}1|s_t{=}m\right)\mu_m\mu_1 + \\
&\quad \gamma_1 \Pr\left(s_{t-k}{=}2|s_t{=}1\right)\mu_1\mu_2 + \gamma_2 \Pr\left(s_{t-k}{=}2|s_t{=}2\right)\mu_2\mu_2 + ... + \gamma_m \Pr\left(s_{t-k}{=}2|s_t{=}m\right)\mu_m\mu_2 + \\
&\quad \vdots \\
&\quad \gamma_1 \Pr\left(s_{t-k}{=}m|s_t{=}1\right)\mu_1\mu_m + \gamma_2 \Pr\left(s_{t-k}{=}m|s_t{=}2\right)\mu_2\mu_m + ... + \gamma_m \Pr\left(s_{t-k}{=}m|s_t{=}m\right)\mu_m\mu_m \\
&= \left(\boldsymbol{\gamma}' \odot \left(\boldsymbol{\mu}'\mathbf{P}^k\right)\right)\boldsymbol{\mu} \\
&= \boldsymbol{\mu}'\mathbf{P}^k\left(\boldsymbol{\gamma} \odot \boldsymbol{\mu}\right)
\end{aligned}$$

and

$$\sum_{i=0}^{T}\theta^{2i} = \frac{1-\theta^{2(T+1)}}{1-\theta^2}$$

as well as

$$\sum_{i=0}^{T-k}\theta^{2i+k} = \frac{\theta^k - \theta^{2(T+1)-k}}{1-\theta^2}$$

the formula simplifies to

$$E\left[Z_t^2\right] = \frac{1-\theta^{2(T+1)}}{1-\theta^2}\boldsymbol{\gamma}'\boldsymbol{\mu}^2 + 2\sum_{k=1}^{T}\left(\frac{\theta^k - \theta^{2(T+1)-k}}{1-\theta^2}\boldsymbol{\mu}'\mathbf{P}^k\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right)\right).$$

If T approaches infinity, one obtains the final formula

$$\begin{aligned} E\left[Z_t^2\right] &= \frac{1}{1-\theta^2}\boldsymbol{\gamma}'\boldsymbol{\mu}^2 + 2\sum_{k=1}^{\infty}\left(\frac{1}{1-\theta^2}\boldsymbol{\mu}'\left(\theta\mathbf{P}\right)^k\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right)\right) \\ &= \frac{1}{1-\theta^2}\boldsymbol{\gamma}'\boldsymbol{\mu}^2 + \frac{2}{1-\theta^2}\left(\boldsymbol{\mu}'\theta\mathbf{P}\left(\mathbf{I}-\theta\mathbf{P}\right)^{-1}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right) \end{aligned}$$

where $\mathbf{I}$ is the identity matrix.

Third noncentral moment: Start by rewriting $E\left[Z_T^3\right]$ as

$$E\left[Z_T^3\right] = E\left[\begin{array}{c} \sum_{i=0}^{T}\theta^i\theta^i\theta^i\mu_{s_{t-i}}\mu_{s_{t-i}}\mu_{s_{t-i}} \\ +\sum_{j=1}^{T}\left(\begin{array}{c} 3\sum_{i=0}^{T-j}\theta^i\theta^i\theta^{i+j}\mu_{s_{t-i}}\mu_{s_{t-i}}\mu_{s_{t-(i+j)}} \\ +3\sum_{i=0}^{T-j}\theta^i\theta^{i+j}\theta^{i+j}\mu_{s_{t-i}}\mu_{s_{t-(i+j)}}\mu_{s_{t-(i+j)}} \end{array}\right) \\ +\sum_{k=1}^{T}\sum_{j=k+1}^{T}\left(6\sum_{i=0}^{T-j}\theta^i\theta^{i+k}\theta^{i+j}\mu_{s_{t-i}}\mu_{s_{t-(i+k)}}\mu_{s_{t-(i+j)}}\right) \end{array}\right] \quad (32)$$

where obviously we have that $i < i+k < i+j$. Assume that in period 0, the state is determined according to the unconditional probabilities, so that

$$\Pr\left(\mu_{s_{t-i}}\mu_{s_{t-(i+k)}}\mu_{s_{t-(i+j)}}\right) = \Pr\left(s_{t-i}|s_{t-(i+k)}\right)\Pr\left(s_{t-(i+k)}|s_{t-(i+j)}\right)\Pr\left(s_{t-(i+j)}\right)$$

holds for all integers i. Hence, we can set $i=0$ and write

$$\Pr\left(\mu_{s_{t-i}}\mu_{s_{t-(i+k)}}\mu_{s_{t-(i+j)}}\right) = \Pr\left(s_t|s_{t-k}\right)\Pr\left(s_{t-k}|s_{t-j}\right)\Pr\left(s_{t-j}\right)$$

and

$$E\left[\mu_{s_{t-i}}\mu_{s_{t-(i+k)}}\mu_{s_{t-(i+j)}}\right]=E\left[\mu_{s_t}\mu_{s_{t-k}}\mu_{s_{t-j}}\right].$$

We will now determine the summation terms of (32) separately. The first term can be rewritten as

$$E\left[\sum_{i=0}^{T}\theta^i\theta^i\theta^i\mu_{s_t}\mu_{s_t}\mu_{s_t}\right]=\boldsymbol{\gamma}'\boldsymbol{\mu}^3\frac{1-\theta^{3(T+1)}}{1-\theta^3}\quad.$$

The second term is equivalent to

$$E\left[\sum_{j=1}^{T}\left(3\sum_{i=0}^{T-j}\theta^i\theta^i\theta^{i+j}\mu_{s_t}\mu_{s_t}\mu_{s_{t-j}}\right)\right]=\sum_{j=1}^{T}E\left[\mu_{s_t}^2\mu_{s_{t-j}}\right]\left(3\sum_{i=0}^{T-j}\theta^{3i+j}\right).$$

An investigation of the expectation $E\left[\mu_{s_t}^2\mu_{s_{t-j}}\right]$ for any integer j yields

$$\begin{aligned}
&E\left[\mu_{s_t}^2\mu_{s_{t-j}}\right]=E\left[\mu_{s_{t+j}}^2\mu_{s_t}\right]=\sum_{i=1}^{m}\sum_{k=1}^{m}\Pr\left(s_{t+j}=i,s_t=k\right)\mu_i^2\mu_k\\
&=\sum_{i=1}^{m}\sum_{k=1}^{m}\Pr\left(s_t=k\right)\Pr\left(s_{t+j}=i|s_t=k\right)\mu_i^2\mu_k\\
&\quad=\gamma_1\Pr(s_{t+j}{=}1|s_t{=}1)\mu_1\mu_1^2+\gamma_2\Pr(s_{t+j}{=}1|s_t{=}2)\mu_2\mu_1^2+\ldots+\gamma_m\Pr(s_{t+j}{=}1|s_t{=}m)\mu_m\mu_1^2+\\
&\qquad\gamma_1\Pr(s_{t+j}{=}2|s_t{=}1)\mu_1\mu_2^2+\gamma_2\Pr(s_{t+j}{=}2|s_t{=}2)\mu_2\mu_2^2+\ldots+\gamma_m\Pr(s_{t+j}{=}2|s_t{=}m)\mu_m\mu_2^2+\\
&\qquad\qquad\vdots\\
&\quad\gamma_1\Pr(s_{t+j}{=}m|s_t{=}1)\mu_1\mu_m^2+\gamma_2\Pr(s_{t+j}{=}m|s_t{=}2)\mu_2\mu_m^2+\ldots+\gamma_m\Pr(s_{t+j}{=}m|s_t{=}m)\mu_m\mu_m^2\\
&=\left(\left(\boldsymbol{\mu}^2\right)'\mathbf{P}^j\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right)\;.
\end{aligned}$$

Thus, we have that

$$
\begin{aligned}
&E\left[\sum_{j=1}^{T}\left(3\sum_{i=0}^{T-j}\theta^{i}\theta^{i}\theta^{i+j}\mu_{s_t}\mu_{s_t}\mu_{s_{t-j}}\right)\right] \\
&\qquad=\tfrac{3}{1-\theta^{3}}\left(\boldsymbol{\mu}^{2}\right)'\left(\sum_{j=1}^{T}\left(\theta^{j}-\theta^{3(T+1)-2j}\right)\mathbf{P}^{j}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right).
\end{aligned}
\tag{33}
$$

On the same lines, one finds that the third term can be expressed as

$$
\begin{aligned}
&E\left[\sum_{j=1}^{T}\left(3\sum_{i=0}^{T-j}\theta^{i}\theta^{i+j}\theta^{i+j}\mu_{s_t}\mu_{s_{t-j}}\mu_{s_{t-j}}\right)\right] \\
&\qquad=\tfrac{3}{1-\theta^{3}}\boldsymbol{\mu}'\left(\sum_{j=1}^{T}\left(\theta^{2j}-\theta^{3(T+1)-j}\right)\mathbf{P}^{j}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}^{2}\right).
\end{aligned}
\tag{34}
$$

Finally, the last term has the representation

$$
\begin{aligned}
&E\left[6\sum_{k=1}^{T}\sum_{j=k+1}^{T}\sum_{i=0}^{T-j}\theta^{i}\theta^{i+k}\theta^{i+j}\mu_{s_t}\mu_{s_{t-k}}\mu_{s_{t-j}}\right] \\
&\qquad=\tfrac{6}{1-\theta^{3}}\left(\sum_{k=1}^{T}\theta^{k}\sum_{j=k+1}^{T}\left(\theta^{j}-\theta^{3(T+1)-2j}\right)E\left[\mu_{s_t}\mu_{s_{t-k}}\mu_{s_{t-j}}\right]\right)
\end{aligned}
$$

where the expectation term is given by

$$
E\left[\mu_{s_t}\mu_{s_{t-k}}\mu_{s_{t-j}}\right]=\sum_{a=1}^{m}\sum_{b=1}^{m}\sum_{c=1}^{m}\Pr\left(s_{t+j}=c|s_{t+j-k}=b\right)\Pr\left(s_{t+j-k}=b|s_t=a\right)\gamma_a\mu_c\mu_b\mu_a\ .
$$

It turns out that this expectation term can be written in matrix form as

$$
E\left[\mu_{s_t}\mu_{s_{t-k}}\mu_{s_{t-j}}\right]=\mathbf{1}_m'\left(\mathbf{A}_1\odot\mathbf{P}^{k}\right)\left(\mathbf{A}_2\odot\mathbf{A}_1\odot\mathbf{P}^{j-k}\right)\mathbf{1}_m
$$

with $\mathbf{A}_1$and $\mathbf{A}_2$ defined by $\mathbf{A}_1:=\boldsymbol{\mu}\otimes\mathbf{1}_m'$ and $\mathbf{A}_2:=\left(\boldsymbol{\mu}\odot\boldsymbol{\gamma}\right)'\otimes\mathbf{1}_m$.

Once T goes to infinity, $E\left[Z_T^3\right]$ thus simplifies to

$$E\left[Z_T^3\right] = E\left[\begin{array}{c} \frac{1}{1-\theta^3}\boldsymbol{\gamma}'\boldsymbol{\mu}^3 \\ +\frac{3}{1-\theta^3}\left(\boldsymbol{\mu}^2\right)'\left(\sum_{j=1}^{\infty}\left(\theta\mathbf{P}\right)^j\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right) \\ +\frac{3}{1-\theta^3}\boldsymbol{\mu}'\left(\sum_{j=1}^{\infty}\left(\theta^2\mathbf{P}\right)^j\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}^2\right) \\ +\frac{6}{1-\theta^3}\sum_{k=1}^{\infty}\mathbf{1}_m'\left(\mathbf{A}_1\odot\left(\theta\mathbf{P}\right)^k\right)\left(\mathbf{A}_2\odot\mathbf{A}_1\odot\sum_{j=k+1}^{\infty}\theta^j\mathbf{P}^{j-k}\right)\mathbf{1}_m \end{array}\right].$$

Using the fact that for any positive integer i, it holds that

$$\sum_{j=1}^{\infty}\left(\theta^i\mathbf{P}\right)^j = \left(\theta^i\mathbf{P}\right)\left(\mathbf{I}-\theta^i\mathbf{P}\right)^{-1},$$

where $\mathbf{I}$ is the identity matrix, and rearranging

$$\sum_{j=k+1}^{\infty}\theta^j\mathbf{P}^{j-k} = \theta^k\sum_{j=1}^{\infty}\left(\theta^j\mathbf{P}^j\right)$$

yields the final formula

$$E\left[Z_T^3\right] = E\left[\begin{array}{c} \frac{1}{1-\theta^3}\boldsymbol{\gamma}'\boldsymbol{\mu}^3 \\ +\frac{3}{1-\theta^3}\left(\boldsymbol{\mu}^2\right)'\left(\theta\mathbf{P}\left(\mathbf{I}-\theta\mathbf{P}\right)^{-1}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}\right) \\ +\frac{3}{1-\theta^3}\boldsymbol{\mu}'\left(\theta^2\mathbf{P}\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\right)\left(\boldsymbol{\gamma}\odot\boldsymbol{\mu}^2\right) \\ +\frac{6}{1-\theta^3}\mathbf{1}_m'\left(\mathbf{A}_1\odot\left(\theta^2\mathbf{P}\right)\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\right)\left(\mathbf{A}_2\odot\mathbf{A}_1\odot\left(\theta\mathbf{P}\right)\left(\mathbf{I}-\theta\mathbf{P}\right)^{-1}\right)\mathbf{1}_m \end{array}\right]. \tag{35}$$

A.2 Steepness of MSI-processes

As stated in the text, disregarding normal errors, the process for ΔZ_t can be written as

$$\Delta Z_t = \mu_{s_t} + (\theta-1)\sum_{i=0}^{\infty}\left(\theta L\right)^i\mu_{s_{t-1}}.$$

Since ΔZ_t has an expectation of zero, the variance of ΔZ_t is equal to

$$E\left[(\Delta Z_t)^2\right] = E\left[\mu_{s_t}^2 + 2\mu_{s_t}(\theta-1)\sum_{i=0}^{\infty}(\theta L)^i \mu_{s_{t-1}} + \left((\theta-1)\sum_{i=0}^{\infty}(\theta L)^i \mu_{s_{t-1}}\right)^2\right].$$

by means of tedious algebra and the results of Appendix A.1, it follows that this variance can be calculated as

$$E\left[(\Delta Z_t)^2\right] = \boldsymbol{\gamma}'\boldsymbol{\mu}^2 + 2(\theta-1)\boldsymbol{\mu}'(\mathbf{I}-\theta\mathbf{P})^{-1}\mathbf{P}(\boldsymbol{\gamma}\odot\boldsymbol{\mu}) + (\theta-1)^2 E\left[Z_t^2\right]$$

where the term $E\left[Z_t^2\right]$ is given by (12). The third moment of ΔZ_t is determined by

$$E\left[(\Delta Z_t)^3\right] = E\left[\begin{array}{c} \mu_{s_t}^3 + 3\mu_{s_t}^2(\theta-1)\sum_{i=0}^{\infty}(\theta L)^i \mu_{s_{t-1}} \\ +3\mu_{s_t}\left((\theta-1)\sum_{i=0}^{\infty}(\theta L)^i \mu_{s_{t-1}}\right)^2 + \left((\theta-1)\sum_{i=0}^{\infty}(\theta L)^i \mu_{s_{t-1}}\right)^3 \end{array}\right].$$

Again by using the results of Appendix A.1, one obtains that this expression is equivalent to

$$\begin{aligned} & E\left[(\Delta Z_t)^3\right] \\ = \; & E\left[\begin{array}{c} \boldsymbol{\gamma}'\boldsymbol{\mu}^3 \\ +3(\theta-1)(\boldsymbol{\mu}^2)'(\mathbf{I}-\theta\mathbf{P})^{-1}\mathbf{P}(\boldsymbol{\gamma}\odot\boldsymbol{\mu}) \\ +3(\theta-1)^2\left[\begin{array}{c} \boldsymbol{\mu}'\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\mathbf{P}(\boldsymbol{\gamma}\odot\boldsymbol{\mu}^2) \\ +2\cdot\mathbf{1}_m'\left(\mathbf{A}_1\odot\left(\mathbf{I}-\theta^2\mathbf{P}\right)^{-1}\mathbf{P}\right)\left(\mathbf{A}_2\odot\mathbf{A}_1\odot(\theta\mathbf{P})(\mathbf{I}-\theta\mathbf{P})^{-1}\right)\mathbf{1}_m \end{array}\right] \\ +(\theta-1)^3 E\left[Z_t^3\right] \end{array}\right], \end{aligned}$$

where $E\left[Z_t^3\right]$ is given by (35) and $\mathbf{A}_1$ and $\mathbf{A}_2$ are again defined by $\mathbf{A}_1 := \boldsymbol{\mu}\otimes\mathbf{1}_m'$ and $\mathbf{A}_2 := (\boldsymbol{\mu}\odot\boldsymbol{\gamma})'\otimes\mathbf{1}_m$.

A.3 Symmetry of MSI-processes with Two States

As claimed in the text, the possible asymmetry of $Z_t = \frac{1}{1-\theta L}\mu_{s_t} + \Upsilon(L)\frac{1}{1-\theta L}\varepsilon_t$ only depends on $\frac{1}{1-\theta L}\mu_{s_t}$. So suppose for a moment that $Z_t = \frac{1}{1-\theta L}\mu_{s_t}$ holds, that the process started in period 1, that the last period is period T and that T approaches infinity. Since for every sequence $\left\{\mu_{s_t}^{(l)}\right\}_{t=1}^{T} = \left\{\mu_{s_1=i}, \mu_{s_2=j}, \ldots, \mu_{s_T=k}\right\}_{t=1}^{T}$ there is a complementary sequence $\left\{\mu_{c,s_t}^{(l)}\right\}_{t=1}^{T} = \left\{\mu_{s_1\neq i}, \mu_{s_2\neq j}, \ldots, \mu_{s_T\neq k}\right\}_{t=1}^{T}$ for $l = 1, 2, 3, ..., 2^{T-1}$, in order to guarantee symmetry around the mean, a sufficient condition is given by the requirement that both sequences have the same probability, which means that

$$\Pr\left(\left\{\mu_{s_t}^{(l)}\right\}_{t=1}^{T}\right) = \Pr\left(\left\{\mu_{c,s_t}^{(l)}\right\}_{t=1}^{T}\right)$$

or equivalently

$$\Pr\left(s_1 = i, s_2 = j, \ldots, s_T = k\right) = \Pr\left(s_1 \neq i, s_2 \neq j, \ldots, s_T \neq k\right)$$

holds, where i, j and k equal either one or two, since we are considering a two-state process. Let $Z_T^{(l)}$ denote the value of Z_T associated with $\left\{\mu_{s_t}^{(l)}\right\}_{t=1}^{T}$ and $Z_{c,T}^{(l)}$ denote the value of Z_T associated with $\left\{\mu_{c,s_t}^{(l)}\right\}_{t=1}^{T}$. Then, independently of the associated probabilities, it is true for all l that $Z_T^{(l)} + Z_{c,T}^{(l)}$ is constant, namely equal to

$$Z_T^{(l)} + Z_{c,T}^{(l)} = \frac{1}{1-\theta L}\left(\mu_1 + \mu_2\right)$$

The mean of Z_T is thus given by

$$E\left[Z_T\right] = \sum_{l=1}^{2^{T-1}} \Pr\left(\left\{\mu_{s_t}^{(l)}\right\}_{t=1}^{T}\right)\left(Z_T^{(l)} + Z_{c,T}^{(l)}\right) = \frac{1}{2}\frac{1}{1-\theta L}\left(\mu_1 + \mu_2\right) \tag{36}$$

Since the mean of Z_T can also be calculated by

$$E\left[Z_T\right] = E\left[\frac{1}{1-\theta L}\mu_{s_T}\right] = \frac{1}{1-\theta L}\left(\gamma_1\mu_1 + \gamma_2\mu_2\right) \tag{37}$$

symmetry requires equality of (36) and (37) which leads to the familiar symmetry condition

$$\gamma_1 = \gamma_2 = \frac{1}{2}$$

that is equivalent to

$$p_{11} = p_{22}.$$

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