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Interest rate risk and the Swiss solvency test

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Non-technical Summary

The Swiss Solvency Test (SST) encompasses the market-consistent valuation of claims and liabilities as well as risk-based capital coverage for insurers. It is, however, virtually impossible to conduct a market-consistent valuation of liabilities because market prices for insurance liabilities are extremely scarce in practice. For this reason, a market-consistent valuation is approximated by calculating the present value of insurance cash flows. The valuation of insurance liabilities therefore essentially depends on the spot interest rates used. As a further consequence, solvency capital backing is also based on the spot interest rates used.

This paper examines how spot rates are modelled for the purpose of risk measurement under the SST standard model and identifies three shortcomings.

First, the SST risk model is based on a large number of interest rate risk factors which necessitate extremely extensive calculation when making a stochastic risk measurement for insurance liabilities and are therefore difficult to compute within a useful timeframe. The SST envisages the modelling and numerical simulation of spot rates for 13 representative maturities for each currency. This is not just time-consuming but also unnecessarily complex.

Second, with respect to risk measurement, the SST assumes normally distributed spot rate changes. This assumption can result in clearly negative and therefore implausible simulations of nominal spot rates, with a particularly serious impact within the currently prevailing low interest rate environment.

Finally, the SST model typically tends toward pro-cyclical capital requirements when capturing interest rate risk. The assumption of normally distributed interest rate changes forces insurance entities to hold more capital in a low interest rate environment than in a high interest rate environment. In actual fact, however, the potential for losses arising from falling interest rates within a low interest rate environment is limited and for this reason higher capital requirements do not appear to make sense in economic terms.

The model put forward by this paper aims to overcome the shortcomings relating to interest rate modelling that are entailed in the SST standard model. First, we use a systematic analysis of the yield curve for spot rates to show how the number of risk factors can be reduced. We propose moving away from individual interest rates for representative maturities, instead modelling the yield curve for all possible maturities directly. In particular, this serves to ensure that numerically simulated yield curves also display the usual em-

pirical form and that the number of interest rate risk factors can be reduced to three per currency.

In addition, it is assumed that spot rates are not normally distributed but instead exhibit a truncated normal distribution pattern, whereby the constraint on interest rate distribution is inferred from the existence of an interest rate lower bound. Such a restriction appears empirically and theoretically reasonable: first, no significant negative nominal spot rates have been observed to date; second, in theoretical terms the lower bound on interest rates is limited by the opportunity cost of holding cash.

Nicht-technische Zusammenfassung

Der Swiss Solvency Test (SST) umfasst die marktnahe Bewertung von Forderungen und Verbindlichkeiten sowie die risikobasierte Kapitalunterlegung für Versicherungen. Gleichwohl ist die marktnahe Bewertung von Verbindlichkeiten nahezu unmöglich, da Marktpreise für versicherungstechnische Verbindlichkeiten in der Praxis kaum zur Verfügung stehen. Sie wird deshalb durch die Barwertbildung von versicherungstechnischen Cashflows approximiert. Die Bewertung von versicherungstechnischen Verbindlichkeiten hängt somit entscheidend von den verwendeten Zinssätzen ab. In weiterer Folge basiert auch die Unterlegung mit Solvenzkapital auf den verwendeten Zinssätzen.

In dieser Arbeit wird die Modellierung von Zinssätzen für die Risikomessung im SST-Standardmodell untersucht, wobei drei Schwachpunkte identifiziert werden.

Erstens baut das SST-Risikomodell auf einer Vielzahl von Zinsrisikofaktoren auf, welche im Rahmen einer stochastischen Risikomessung für versicherungstechnische Verbindlichkeiten äußerst rechenintensiv und somit innerhalb nützlicher Frist nur schwerlich zu bewältigen sind. Der SST sieht die Modellierung und numerische Simulation von Zinsen für 13 repräsentative Laufzeiten für jede Währung vor, was nicht nur aufwändig, sondern auch komplex ist.

Zweitens unterstellt der SST für die Risikomessung normalverteilte Zinsänderungen. Diese Annahme kann zu deutlich negativen und somit nicht plausiblen Simulationen von nominalen Zinsen führen, was sich in dem zurzeit vorherrschenden Niedrigzinsumfeld besonders gravierend auswirkt.

Schlussendlich führt der SST typischerweise für die Zinsrisikomessung zu prozyklischen Kapitalanforderungen. Die Annahme normalverteilter Zinsveränderungen zwingt Versicherungsunternehmen im Tiefzinsumfeld dazu, mehr Eigenkapital als im Hochzinsumfeld zu halten. Tatsächlich ist im Tiefzinsumfeld das Verlustpotential aus fallenden Zinsen jedoch begrenzt, weshalb eine erhöhte Kapitalanforderung ökonomisch nicht sinnvoll erscheint.

Das in dieser Arbeit vorgeschlagene Modell verfolgt das Ziel, die Mängel der Zinsmodellierung im SST-Standardmodell zu beseitigen. Zunächst wird anhand einer systematischen Analyse der Zinsstrukturkurve von Zinsen gezeigt, wie die Anzahl an Risikofaktoren reduziert werden kann. Es wird vorgeschlagen, anstatt Zinssätze für repräsentative Laufzeiten direkt die Zinskurve für alle denkbaren Laufzeiten zu modellieren. Dies garantiert insbesondere, dass auch numerisch simulierte Zinskurven die übliche empirische Form aufweisen

und die Anzahl der Zinsrisikofaktoren auf drei pro Währung reduziert wird.

Zusätzlich wird angenommen, dass Zinssätze nicht normalverteilt, sondern gestutzt normalverteilt sind, wobei die Beschränkung der Zinsverteilung aus der Existenz einer Zinsuntergrenze gefolgert wird. Eine solche Restriktion erscheint theoretisch und empirisch sinnvoll. Zum einen wurden signifikant negative, nominale Zinssätze bisher nicht beobachtet, zum anderen ist die Zinsuntergrenze theoretisch durch die Opportunitätskosten der Bargeldhaltung begrenzt.

Bundesbank Discussion Paper No 41/2013

Interest Rate Risk and the Swiss Solvency Test*

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Abstract

In this paper, we present a new approach to measuring interest rate risk for insurers within the Swiss Solvency Test, which overcomes the shortcomings of the standard model. The standard model of the Swiss Solvency Test is based on more interest rate risk factors than are actually needed to capture interest rate risk, it allows for significantly negative interest rates and it tends toward procyclical solvency capital requirements. Our new approach treats interest rate risk with direct reference to the underlying term structure model and interprets its parameters as a canonical choice of the relevant interest rate risk factors. In this way, the number of interest rate risk factors is substantially reduced and interest rate risk measurement is linked to the term structure model itself. The consideration of empirical interest rate data and the acceptance of the economical implausibility of persistently negative interest rates significantly below the cost of holding cash motivate the introduction of a truncated Gaussian process to simulate innovation in the future development of the parameters of the underlying term structure model. In a natural way this leads to mean-reverting interest rate behaviour and to countercyclical solvency capital requirements.

Keywords: interest rate risk, yield curve, truncated Gaussian process, Swiss Solvency Test
JEL: C51, C58, G22, G28

*Contact address: Armin Eder, Helvetia Insurance, Dufourstrasse 40, CH-9001 St. Gallen, Switzerland, Tel +41 58 280 51 73, E-mail armin.eder@helvetia.ch. Numerous people have contributed to this paper. The authors would especially like to thank Christoph Memmel and Michael Pfaffermayr for their contributions. Discussion Papers represent the authors' personal opinions and do not necessarily reflect the views of the Deutsche Bundesbank, Helvetia Group or its staff.

1 Introduction

As 2011 began, and after a five-year transition period, the Swiss Solvency Test (SST) came into force requiring insurance undertakings to meet regulatory capital requirements. The quantitative aspects of this new regulatory framework are based on market-consistent valuation of assets and liabilities and on solvency capital requirements which reflect the risk profile of the undertakings' balance sheets and the underwritten business.

Solvency is measured by comparing the required amount of solvency capital with the amount of available capital. Within the framework of the SST, available capital is referred to as Risk Bearing Capital (RBC). The RBC can be interpreted as the undertaking's capacity to write new business and to absorb future losses. At a given point in time, the RBC is calculated on the basis of the balance sheet of a company.[1] The RBC is derived as the sum of the net asset value (NAV) of the *market-consistent* balance sheet and – if existing – of subordinated hybrid capital less possible capital deductions such as anticipated future dividends.

For the assessment of an undertaking's solvency position under the SST, the *existing* capacity of a company to absorb losses and to cope with risks in general needs to be confronted with a quantity that captures *what could go wrong in the future* with the company within a certain period of time. The qualitative question is quantitatively specified by the definition of the required amount of solvency capital, the so called Target Capital (TC), as the Expected Shortfall (ES) of the undertaking's aggregate loss distribution at the one-percent quantile after a period of one year ($ES_{0.01}$). An insurance undertaking meets solvency if the RBC exceeds the TC, i.e. if $RBC \geq TC$. Without too much formal rigour this definition can be understood as follows: the SST requires entities to hold enough available capital that, out of one hundred companies with a solvency coverage ratio $\geq 100\%$ at $t = 0$, the average number of companies defaulting within one year is less than one.[2]

The determination of the aggregate loss distribution of a given undertaking after one year is at the centre of the calculation of the TC. While the RBC is observable at any given point in time, $RBC_{t=0} := RBC_0$, the value of RBC at $t = 1$ is unknown at $t = 0$. The RBC at $t = 1$ is therefore a random variable. From the distribution of $RBC_{t=1}$, the

[1] Modern insurance regulation typically requires a market-consistent valuation of insurance liabilities which in turn is often achieved by means of stochastic valuation. However, the RBC is given at any point in time once the expected time value of management rules and policyholder behaviour was analysed.

[2] For further details the reader is referred to Swiss Financial Market Supervisory Authority (2006) and Keller and Luder (2004).

aggregate loss distribution $\Delta RBC_{t=1} = v \cdot RBC_{t=1} - RBC_0$ of the undertaking can be calculated and the ES at the one-percent quantile can be read.[3] The factor $v = 1/(1+r_1)$ discounts the $RBC_{t=1}$ to period $t = 0$ with a risk-free one-year spot rate. In this paper, it is assumed that losses arise because of adverse deviations from expectations about the future. Within the framework of the SST, losses may result from changes in the market-consistent valuation of assets and liabilities, from deviations from the expected result of underwriting insurance business and counterparty defaults. These losses need to be recognised in the aggregate loss distribution. Stated slightly differently, the SST considers a set of *risk factors* as the sources of possible losses in available capital. These risk factors include financial market risks, technical underwriting risks and counterparty default risks. One of the most prominent sources of risk is the volatility of interest rate levels. Interest rate risk refers to changes in risk-free interest rates. In the SST, risk-free interest rates are derived from government bond yields. The role and the measurement of the associated interest rate risk is the subject of this paper.

Interest rates are of twofold relevance within the SST. Firstly, the importance of interest rates and their term structure derives from the simple fact that there is no market value for insurance liabilities, at least not for all kinds of liability.[4] Yet, the SST requires the assignment of a market value to insurance liabilities (MVL). This problem is circumvented operationally by approximating the market value of insurance liabilities through the sum of the best estimate of insurance liabilities (BEL) and a market value margin (MVM): $MVL \approx BEL + MVM$. Both the BEL and the MVM depend on the underlying interest rates and their term structure.[5] Therefore, shifts in interest rates affect the SST through its impact on the valuation of insurance liabilities. However, this kind of RBC volatility is determined by observable interest rate changes from one point in time to another.

Secondly, and of more relevance for the present paper, risk-free interest rates themselves

[3] Strictly speaking, the SST loss distribution constitutes an aggregation of a continuous loss distribution and losses generated by specific scenarios with predefined occurrence probabilities.

[4] Liabilities deriving from pure linked business may be valued directly by the market value of the related assets.

[5] It might be argued that the BEL and the MVM are communicating vessels in terms of the impact of the underlying interest rates and that it is only the sum of both which matters (Keller, Gisler, and Wüthrich, 2011). However, calculating of the MVM is a delicate problem in its own right and practical experience indicates that both quantities are not two sides of the same coin to an extent that would the basis for a theory.

In the SST, the RBC is calculated using only the BEL rather than the full market value of liabilities MVL. The MVM becomes part of the TC. Instead of reducing the RBC by the MVM, the SST increases the TC by the MVM, which is conceptually somewhat fuzzy. Contrary to the SST, the new European Solvency II explicitly treats both BEL and MVM as components of the market value of insurance liabilities; hence, the existence of an MVM leads directly to a reduction of available capital rather than to an increase in required capital.

and the modelling of future risk-free interest rates typically are of great significance for the amount of required capital. It is natural that an undertaking's exposure to risk-free interest rates depends on that company's business model and asset-liability policy. For instance, a company writing short-tailed P&C business is obviously much less exposed to interest rate risk than a company with, say, a moderate quality ALM policy engaged in traditional life business. For the latter, interest rate risk may easily become the dominant driver of TC.

As stated before, in this paper we are concerned with measuring interest rate risk and calculating of the associated solvency capital. In this context, we explain which properties an economically sensible solvency regulation should have in our opinion and we briefly outline what we consider to be the shortcomings of the SST standard model. This synopsis is followed by a detailed presentation of our approach to model interest rate risk.

2 The SST standard model: an overview

As a basis for quantitatively determining the solvency position of an undertaking, the Swiss Financial Market Supervisory Authority FINMA encourages and – depending on the scale and complexity of the regulated undertaking – even demands the development of internal risk models. At the same time, FINMA also provides a standard model for quantifying the TC, in particular for calculating the required capital associated with potential losses due to changes in risk-free interest rate levels.

2.1 The standard model in a nutshell

When the SST was initiated, the standard model for financial market risks started out with a loss distribution that was based on a multilinear approximation of the $RBC_{t=1}$ at RBC_0 with respect to market risk factors $x_{1,t}, \ldots, x_{n,t}$. The RBC at $t = 1$ read

$$RBC(x_{1,t=1}, \ldots, x_{n,t=1}) = RBC(x_{1,t=0} + \Delta x_1, \ldots, x_{n,t=0} + \Delta x_n), \tag{1}$$

which was approximated by a linear function as follows (with all derivatives calculated at $t = 0$):

$$RBC(x_{1,t=1}, \ldots, x_{n,t=1}) \approx RBC(x_{1,t=0}, \ldots, x_{n,t=0}) + \sum_{i=1}^{n} \frac{\partial RBC}{\partial x_i} \cdot \Delta x_i. \tag{2}$$

The marginal distributions of each market risk factor x_i were assumed to be Gaussian distributions. The entire loss distribution for all market risks under consideration was modelled as a multivariate Gaussian distribution which had the advantage that the ES could be calculated analytically with random realisations of all risk factors being drawn as one vector from a multivariate Gaussian distribution as shown in Equation (3).

$$\Delta \mathbf{x} \sim N(\mathbf{0}, \mathbf{\Sigma}) \tag{3}$$

$\Delta \mathbf{x}$ indicates the return or the change of the risk factor, respectively, i.e. the first difference $\Delta \mathbf{x} = \mathbf{x}_{t=1} - \mathbf{x}_0$.[6] $N(\mathbf{0}, \mathbf{\Sigma})$ represents the multivariate normal distribution with mean zero and covariance matrix $\mathbf{\Sigma}$.[7] The SST standard model uses 120 monthly observations to calculate the monthly volatilities of the risk factors and scales these volatilities with $\sqrt{12}$ in order to obtain annual volatilities.

As an example, the parametrisation of representative interest rate risk factors as used for the SST as of year end 2011 is shown in Table 1 below.

Time to maturity	Zero rate $x_{t=0}$	Volatility	$ES_{0.01}(x_{t=1})$	Observed minimum
1 year	15.5	60.3	-145.2	1.2
5 years	19.9	58.7	-136.5	-5.0
10 years	74.0	55.2	-73.1	53.0
30 years	163.9	53.3	21.8	105.6

Table 1: Parametrisation of the marginal distribution of four interest rate risk factors. Expected shortfall estimate at the one percent level. Numbers are quoted in basis points.

In the currently applicable version of the standard model, the linear expansion of $RBC_{t=1}$ was replaced by a second order Taylor expansion in all market risk factors at RBC_0. This so called Delta-Gamma approximation adds to Equation (2) quadratic and mixed partial derivatives of the RBC with respect to market risk factors. Although this modification takes account of convexity effects in the loss distribution that were not addressed in the original standard model, the shortcomings associated with the marginal distributions of the risk factors themselves were not covered. These will be discussed in the next section.

[6] The SST standard model uses log-differences for non-interest rate risk factors, and first differences of interest rate risk factors. Since we deal with interest rate risk we only consider first differences.

[7] As usual, bold symbols indicate vectors or matrices.

2.2 Shortcomings of the standard model

The standard model undoubtedly has the advantage of being simple and immediately comprehensible. On the other hand, the example of the expected shortfall in Table 1 demonstrates that the standard model leads to quite disputable results, in particular in a low interest rate environment such as the one prevailing at present. An expected shortfall of roughly -150 basis points for the one year interest rate seems remarkably low and is, from an economic perspective, rather hard to justify.

In general, we are convinced that the standard model has various shortcomings, in particular with respect to the treatment of risk-free interest rates. These shortcomings can be summarized as follows:

- Allowance for significantly negative interest rates: as illustrated in Table 1, negative interest rates that are hard to reconcile with economic reality (even for Switzerland and Germany at the peak of the European public debt crisis) are possible.
- Procyclicity of capital requirement: in the standard model, the relation between the level of risk-free interest rates and the required solvency capital tends to be procyclical.[8] When evaluating the interest rate risk of a given undertaking with positive duration of the RBC in the standard model, one finds that the company's TC in a low-interest rate environment essentially stays the same as would be in the case in an environment of higher interest rates.[9] [10] In our opinion, this result is problematic as in a low-interest rate environment the capital requirement *should drop significantly* as the down-side potential of already low interest rates is very limited and might ultimately be determined by the cost of holding cash. On the other hand, the down-side potential of high interest rates should be reflected in higher capital requirements. Hence, the standard model leads to a capital requirement for interest rate risk in a low-interest rate environment that is far too high relative to the capital requirement within a high-interest rate environment. This behaviour is attributable to the use of Gaussian distributions for modelling interest rate risk, which at the same time leads to a robust, but economically unjustified capital requirement. In

[8] Typically, the liabilities of insurance undertakings are *longer* than their assets, therefore the solvency capital requirement is driven by low interest rates.

[9] Rather than using the standard Macauley definition with a negative sign, we define the duration of the RBC as the partial derivative of the RBC with respect to the interest rate. Consequently, a positive duration of the RBC implies losses in RBC in case of decreasing interest rates.

[10] In fact, it is true that the TC would actually increase because of the convexity of the RBC, i.e. due to increasing duration in line with decreasing interest rates.

short, we believe that companies should be required to hold *more capital in good times and less in bad times.*

- Multitude of risk factors: the standard model incorporates significantly more interest rate risk factors than are actually needed. This is due to the fact that the functional dependence between the term structure of interest rates and the underlying models used to produce these term structures is not reflected in the standard model at all. When combined with stochastic valuation techniques for life business (similar to market-consistent embedded value calculations), the presence of more interest rate risk factors than necessary leads to costly and time consuming numerical simulations which are not essential for modelling interest rate risk.

In this paper, we address the aforementioned shortcomings of the SST standard model and present ways to improve measuring interest rate risk. Under the current regulatory framework of the SST, these improvements would have to be implemented within a company-specific internal risk model.

3 Methodology towards a new interest rate risk model

The development of a generic risk model involves three important steps. The first step consists in identifying the risk drivers. The second step would be to set up a model that describes the evolution of these risk drivers over time. While the third step would be the development of a model parameter estimator. This section concentrates on the realisation of this programme with respect to interest rate risk. First, we will isolate the interest rate risk factors. After that, we will introduce the risk model and the parameter estimation.

3.1 Interest rate risk factors

In order to introduce interest rate risk within a risk model, the term structure of interest rates – the functional relationship between time to maturity and the spot rate – needs to be specified. The SST standard model uses thirteen spot rate maturity buckets to model the change of the shape of the yield curve (for each currency in which the undertaking has a material investment). Setting up a risk model in line with the standard model entails the extensive task of modelling all thirteen buckets separately. However, we argue that modelling the spot rates directly is inefficient and may lead to badly behaving yield

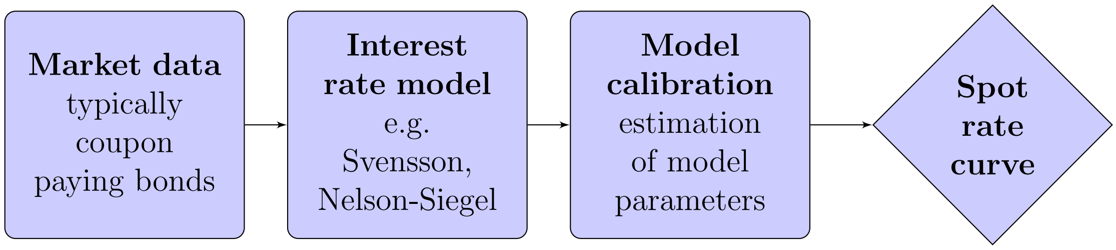


Figure 1: Data generating process of spot rate curves

curves and ignores the functional dependencies across the buckets. An alternative involves reconsidering the data generating process of spot rate curves directly.

3.1.1 The data generating process of spot rate curves

Typically, we do not observe long-term zero coupon yields directly, nor do we have enough market data on coupon-paying bonds to bootstrap the yield curve. Thus, a term structure model is needed to extract the spot rate curve from the available market data. For example, the Swiss National Bank's (SNB) spot rates are calculated using the Svensson (1994) model which is calibrated on the basis of market prices and cash-flow patterns of coupon paying bonds (Müller, 2002). The Swiss Financial Market Supervisory Authority uses exactly these interest rates for valuation purposes within the SST.

The Svensson (1994) model is a parsimonious four factor interest rate model and is widely used by central banks such as the ECB or the Deutsche Bundesbank. It is a variant of the Nelson-Siegel model (Nelson and Siegel, 1987). In the standard model of the SST, SNB spot rates are used for the purpose of valuation and to model Swiss Francs risk-free interest rate risk. The Svensson model assumes that the instantaneous forward rate is of the following functional form:[11]

$$f(\tau, \boldsymbol{\theta}_t) = c_{1,t} + c_{2,t} \cdot \left[e^{-\tau\lambda_{1,t}}\right] + c_{3,t} \cdot \left[\tau\lambda_{1,t} e^{-\tau\lambda_{1,t}}\right] + c_{4,t} \cdot \left[\tau\lambda_{2,t} e^{-\tau\lambda_{2,t}}\right] \tag{4}$$

In this expression, the parameter vector $\boldsymbol{\theta}_t$ reads $\boldsymbol{\theta}_t = [c_{1,t}, c_{2,t}, c_{3,t}, c_{4,t}, \lambda_{1,t}, \lambda_{2,t}]$ and the time to maturity is given by τ. The resulting functional form of the spot rate curve is

[11] The instantaneous forward rate $f(\tau)$ is obtained from convergence of the maturity of the contract to zero, i.e. $\lim_{\hat{\tau} \to 0} f(\tau, \hat{\tau}) = f(\tau)$.

then given by:[12]

$$r(\tau, \boldsymbol{\theta}_t) = c_{1,t} + c_{2,t} \cdot \left[\frac{1 - e^{-\tau\lambda_{1,t}}}{\tau\lambda_{1,t}}\right] + c_{3,t} \cdot \left[\frac{1 - e^{-\tau\lambda_{1,t}}}{\tau\lambda_{1,t}} - e^{-\tau\lambda_{1,t}}\right] + \\ + c_{4,t} \cdot \left[\frac{1 - e^{-\tau\lambda_{2,t}}}{\tau\lambda_{2,t}} - e^{-\tau\lambda_{2,t}}\right] \quad (5)$$

The production of the spot rate yield curve is fully specified by the data generating process sketched in Figure 1 and the mathematics in Equation (5). From this it is evident that the vector $\boldsymbol{\theta}_t$ represents a natural set of risk factors and is better suited for the measurement of interest rate risk than any set of interest rate levels for individual maturities or maturity buckets. The main reason for using $\boldsymbol{\theta}_t$ as risk factors for our risk model is the fact that the entire space of possible future interest rates and yield curves is included in Equation (5). Obviously, it is neither necessary nor even appropriate to model thirteen spot rate buckets if all future interest rates can be produced by no more than six parameters. There would be a limitation of this conclusion if the spot rates were derived from a very different term structure model than the Svensson model (or in general other than the Nelson-Siegel model class). However, many empirical examinations of the Nelson-Siegel class of term structure models have shown that it is flexible enough to approximate any shapes associated with yield data (De Pooter, 2007). This suggests that the interest rate risk factor space per currency of the SST standard model – $\mathbb{R}^{13}$ – is oversized and can be reduced to $\mathbb{R}^6$ per currency without any loss of generality.

To summarize, our analysis of the data generating process of spot rate curves demonstrates that the risk manager is well advised to deviate from the SST standard model. Alternatively, she might work with the Svensson model and interpret the vector $\boldsymbol{\theta}_t$ as the vector of new risk factors rather than set up an interest rate model based on thirteen spot rate buckets. Such an approach guarantees that the simulated yield curves will display features that are typically observed in yield curve data: monotonicity and both hump and S-shapes. In short: when thirteen spot rate risk buckets are modelled separately, the simulated curve might be neither smooth nor correspond to the shape of a spot rate curve.

12 Note that $r(\tau) = \frac{1}{\tau}\int_0^\tau f(s)ds$.

3.1.2 Simplifying the Svensson model

A closer inspection of Equation (5) highlights a potential pitfall. If $\lambda_{1,t}$ and $\lambda_{2,t}$ are rather similar, the individual values of the parameters $c_{3,t}$ and $c_{4,t}$ cannot be identified – only the sum of $c_{3,t}$ and $c_{4,t}$ is empirically accessible. Additionally, multicollinearity problems arise when the decay parameters $\lambda_{1,t}$ and $\lambda_{2,t}$ take on extreme values. For example, when the decay parameters approach zero, the factors multiplying $c_{2,t}$ and $c_{3,t}$ – the so-called factor loadings – will be highly collinear (De Pooter, 2007).[13] As a consequence, the parameters may be estimated to have large values with offsetting signs as was reported by Gimeno and Nave (2006). We circumvent this problem by assuming that $\lambda_{1,t} = \lambda_{2,t} = \lambda$. This leads to the classical Nelson and Siegel (1987) model, specified in Equation (6).[14]

$$r(\tau, \boldsymbol{\theta}_t) = c_{1,t} + c_{2,t} \cdot \left[\frac{1 - e^{-\tau\lambda}}{\tau\lambda}\right] + c_{3,t} \cdot \left[\frac{1 - e^{-\tau\lambda}}{\tau\lambda} - e^{-\tau\lambda}\right] + \nu_{t,\tau} \tag{6}$$

Alternatively, Equation (6) can be expressed in terms of factor loadings $l_i(\tau)$ or by the model matrix $\mathbf{L}(\tau) = [l_1(\tau), l_2(\tau), l_3(\tau)]$:

$$r(\tau, \boldsymbol{\theta}_t) = c_{1,t} \cdot l_1(\tau) + c_{2,t} \cdot l_2(\tau) + c_{3,t} \cdot l_3(\tau) + \nu_{t,\tau} \tag{7}$$

$$= \mathbf{L}(\tau) \cdot \boldsymbol{\theta}_t + \nu_{t,\tau} \tag{8}$$

The assumption of $\lambda_{1,t} = \lambda_{2,t}$ certainly decreases the model flexibility, but the in-sample analysis in section 5.1 demonstrates that the model fit with Nelson-Siegel is perfectly adequate for risk measurement purposes.[15] Unlike Equation (5), Equation (6) contains an error term $\nu_{t,\tau}$ that depends on the point in time and the time to maturity. The reason for this new error term is obvious. The Svensson model is the original data generating process of the spot rate curve, hence the model produces a perfect fit. The Nelson-Siegel model, on the other hand, is based on fewer factors than the Svensson model and therefore cannot reproduce the spot rates exactly.

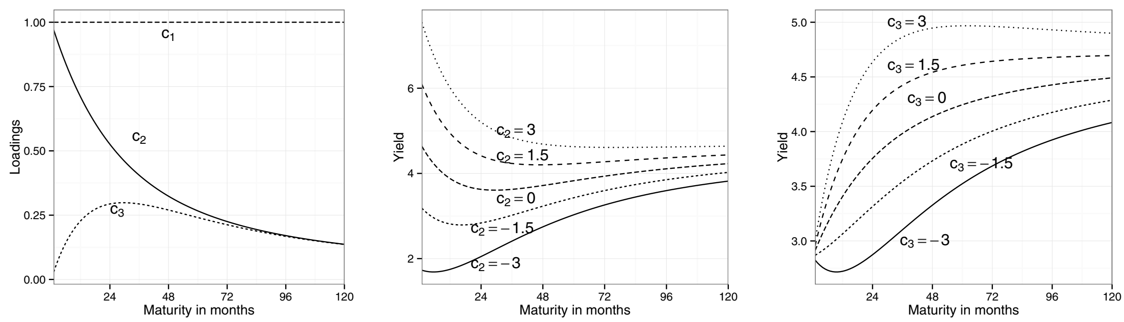
The parameter λ determines the maximum of the factor loading coefficient $l_3(\tau)$ of the parameter $c_{3,t}$. Diebold and Li (2006) propose setting $\lambda = 0.0609$. Although the parametrisation of spot rates in Equation (6) with $c_{i,t}$ is wholly linear, it is highly flexible. Changes in the parameter c_1 shift the whole curve up and down meaning that c_1 can be inter-

13 When the decay parameter approaches zero, the factor loading on $c_{2,t}$ is $\lim_{\lambda \to 0} \left[\frac{1-e^{-\tau\lambda}}{\tau\lambda}\right] = 1$. Thus the parameters $c_{1,t}$ and $c_{2,t}$ cannot be reliably identified.

14 For the sake of simplicity, we denote the Nelson-Siegel parameters using the same symbol $\boldsymbol{\theta}_t$ that we already used for the Svensson model.

15 For a detailed discussion on the model fit of the Nelson-Siegel class of term structure models, see De Pooter (2007).

preted as the level of the yield curve. Parameter c_2 describes the slope of the yield curve, and parameter c_3 influences the curvature. Both parameters are explored further in a comparative static analysis (see Figure 2).



(a) Factor loadings (b) Parameter c_2 (c) Parameter c_3

Figure 2: Partial derivatives of the Nelson-Siegel model with respect to parameters c_k, $k = 1, 2, 3$; these partial derivatives are the so-called factor loadings. Comparative static analyses of parameters c_2 and c_3. All other parameters are held constant.

Variation of c_2 affects the steepness of the yield curve (subfigure 2(b)). A positive factor implies an inverse yield curve while a negative factor produces an upward sloping term structure of interest rates. Changes in the parameter c_3 (subfigure 2(b)) alter the curvature of the yield curve. The hump and S-shapes, typically observed with yield data, are fitted by c_3. In short, the Nelson-Siegel coefficients can be interpreted as yield curve level, slope and curvature. Hereafter, we will regard these coefficients as the natural choice of risk factors for measuring interest rate risk.

Subfigure 2(a) explores the functional form of the factor loadings. These factor loadings play a crucial role when calculating the partial derivatives of the RBC with respect to the Nelson-Siegel coefficients. Whilst changes in the level of the yield curve (c_1) alter the whole yield curve to the same extent, variations in the slope (c_2) have a decreasing impact on interest rates. The partial derivative of $r(\tau, \boldsymbol{\theta}_t)$ with respect to the curvature parameter c_3 peaks at approximately 2.5 years (as superimposed by setting $\lambda = 0.0609$) and consequently decreases towards longer maturities.

3.2 Expansion of the *RBC*

Having identified the three parameters of the Nelson-Siegel model as the most natural set of risk factors, $\boldsymbol{\theta}_t = [c_{1,t}, c_{2,t}, c_{3,t}]$, we now have to provide a valuation function for

$RBC(\boldsymbol{\theta}_t)$ to determine its variation over time, i.e. to determine the distribution of future gains and losses as a function of the risk factors. Following the SST standard model in this respect, we propose to use a second order Taylor expansion in $\boldsymbol{\theta}_t$ as the valuation function.[16]

The suggested Taylor expansion of $RBC_{t=0}$ may be applied in two ways. Firstly, it can be used to calculate the solvency capital requirement for interest rate risk by translating simulated Nelson-Siegel parameters at $t = 1$ into the Expected Shortfall of gains and losses of an undertaking due to shifting interest rates. Secondly, the Taylor expansion of $RBC_{t=0}$ can be used to estimate the new value of RBC at an instant Δt later, $RBC_{t=\Delta t}$, by taking into account the factual changes in spot rates that have occurred in the market over a period Δt since the last calculation of the Risk Bearing Capital. For both applications, the link between observed spot rates and the Nelson-Siegel parameters is a prerequisite.[17]

When we consider the case of risk-free interest rates for one currency, the – at any point in time – measurable quantity RBC_t can be expressed by two sets of risk factors and consequently two different functions $RBC_{1,t}$ and $RBC_{2,t}$.[18] The first function $RBC_{1,t}$ refers to thirteen spot rates or spot rate buckets as risk factors, $\boldsymbol{r_t} = [r_{1,t}, \ldots, r_{13,t}]$. In our approach, the same quantity RBC is expressed by function $RBC_{2,t}$ using the three parameters $c_{1,t}$, $c_{2,t}$ and $c_{3,t}$:

$$\begin{aligned} RBC_t &= RBC_{1,t}(r_{1,t}, \ldots, r_{13,t}), \\ &= RBC_{2,t}(c_{1,t}, c_{2,t}, c_{3,t}). \end{aligned} \tag{9}$$

The partial derivatives with respect to the two sets of risk factors can easily be transformed into each other; the sensitivity of the Risk Bearing Capital with respect to any of the Nelson-Siegel coefficients $\boldsymbol{\theta}_t$ is a weighted sum of the RBC sensitivities with respect to spot rates, where the respective weights are the factor loadings of the Nelson-Siegel model:[19]

$$\frac{\partial RBC_{2,t}(\boldsymbol{\theta}_t)}{\partial c_k} = \frac{1}{T} \int_0^T \frac{\partial RBC_{1,t}(\boldsymbol{r}_t)}{\partial r(\tau, \boldsymbol{\theta}_t)} \cdot \frac{\partial r(\tau, \boldsymbol{\theta}_t)}{\partial c_k} \, \mathrm{d}\tau \tag{10}$$

[16] It should be self-evident that one could also use other valuation functions. Recently with the advent of Solvency II, several other approximation techniques have been discussed.

[17] Remember that the Nelson-Siegel model is a simplification of the Svensson spot rate model that was the starting point for our analysis. Therefore, the link between spot rate sensitivities and sensitivities with respect to the three Nelson-Siegel parameters can only be an approximate, but highly accurate.

[18] Naturally, the RBC_t is determined from a market-consistent balance sheet.

[19] This holds true if and only if no cross derivatives of the RBC with respect to spot rates exist.

Expressed in terms of Nelson-Siegel factor loadings, Equation (10) reads

$$\frac{\partial RBC_{2,t}(\boldsymbol{\theta}_t)}{\partial c_k} = \frac{1}{T}\int_0^T \frac{\partial RBC_{1,t}(\boldsymbol{r}_t)}{\partial r(\tau, \boldsymbol{\theta}_t)} \cdot l_k(\tau)\, \mathrm{d}\tau. \tag{11}$$

Another approach to calculate the partial derivative of the RBC with respect to $\boldsymbol{\theta}_t$ would be scenario analysis. This approach is especially promising for life insurance undertakings which use computationally burdensome and costly stochastic valuation techniques and are obliged to assess interest rate risk. Consequently, calculating only a few interest rate scenarios instead of dozens of sensitivity runs is advisable. For example, the effect of a shift of the entire yield curve by ±100 basis points, i.e. $l_1 \cdot (\pm 100)$ basis points, is calculated and the partial derivative of the RBC with respect to c_1 is subsequently calculated numerically.

3.3 The risk model

In the previous subsections we isolated the relevant risk factors and discussed one possible approximation of the valuation function of the RBC. The final component of a risk model is the stochastic process of the risk factors. The SST standard model assumes that the risk factors follow a random walk,

$$c_{1,t+1} = c_{1,t} + \epsilon_{1,t} \tag{12}$$

$$c_{2,t+1} = c_{2,t} + \epsilon_{2,t} \tag{13}$$

$$c_{3,t+1} = c_{3,t} + \epsilon_{3,t} \tag{14}$$

or, in the more convenient matrix notation,

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \boldsymbol{\epsilon}_t. \tag{15}$$

Typically, $\boldsymbol{\epsilon}_t$ is modelled by a multivariate Gaussian distribution with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. This has two important implications. Firstly, if $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are assumed to be constants, i.e. without any sampling errors, it automatically follows that $\boldsymbol{\theta}_{t+1}$ is Gaussian as well.[20] Secondly, since the spot rates are weighted averages of the term structure coefficients $\boldsymbol{\theta}_{t+1}$,

[20] Note that the $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated from the data and are subject to sampling errors. As a result, $\boldsymbol{\theta}_{t+1}$ is t-distributed. Since the t-distribution converges very fast to the Gaussian, it is reasonable to think of $\boldsymbol{\theta}_{t+1}$ as Gaussian as well. In practice the sampling error is often disregarded which may result in too narrow distributions of the predictions. However, the effect may be minor compared with other sources of error (Chatfield, 1993, 2000).

they belong to the same class of distributions.[21]

However, the assumption of the error terms' normality may be inappropriate. From a theoretical economical perspective, the lower bound of interest rates might be determined by the cost of holding cash. Consequently, a symmetric distribution of interest rates may be misleading, especially in low interest environments. In fact, highly negative spot rates – such as implied by a Gaussian – have never been observed. Yet it is also true that for medium and long-term interest rates the hypothesis of a Gaussian distribution cannot always be rejected (see for example Table 2).

We address this issue by assuming that the error vector $\boldsymbol{\epsilon}_t$ follows a *truncated Gaussian* distribution:

$$\boldsymbol{\epsilon}_t \;\sim\; TN_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma} \mid \Omega_t). \tag{16}$$

$\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the location vector and the scale matrix of the truncated multivariate Gaussian distribution. The truncation of the Gaussian is mediated by conditioning on the information set given at time t. Qualitatively speaking, the truncation of the error terms' Gaussian is introduced by taking into account the existence of economically reasonable lower bounds on interest rates. Once any lower bound on interest rates is defined at $t = 0$, it defines the accessible area of the ϵ-space, Ω_t for any given time t.

In this paper, the truncation is introduced via conditioning on the instantaneous forward rate defined in Equation (4): the instantaneous forward rate $f(\boldsymbol{\tau}, \boldsymbol{\theta}_{t+1})$ needs to be at least equal to $\hat{f}(\boldsymbol{\tau})$, the original lower bound of the instantaneous forward rate, at any point in time in the future:[22]

$$f(\boldsymbol{\tau}, \boldsymbol{\theta}_{t+1}) \geq \hat{f}(\boldsymbol{\tau}). \tag{17}$$

We translate the condition of Equation (17) into a definition of the accessible area of the ϵ-space, Ω_t, by referring to the random walk of the risk factors $\boldsymbol{\theta}_{t+1}$:

$$c_{1,t} + \epsilon_{1,t} + (c_{2,t} + \epsilon_{2,t}) \cdot \left[e^{-\tau\lambda}\right] + (c_{3,t} + \epsilon_{3,t}) \cdot \left[\tau\lambda e^{-\tau\lambda}\right] \geq \hat{f}(\tau). \tag{18}$$

Introducing the matrix $\mathbf{A}$ of dimension $(\tau, 3)$ of the instantaneous forward factor loadings, $\mathbf{A} = [\mathbf{1}, \left(e^{-\boldsymbol{\tau}\lambda}\right), \left(\boldsymbol{\tau}\lambda e^{-\boldsymbol{\tau}\lambda}\right)]$, we formulate the truncation condition as follows:

$$\mathbf{A} \cdot \boldsymbol{\epsilon}_t \geq \hat{f}(\boldsymbol{\tau}) - \mathbf{A} \cdot \boldsymbol{\theta}_t. \tag{19}$$

[21] If one assumes that $\boldsymbol{\epsilon}_t$ is multivariate Gaussian, one simply ends up with the SST standard model, but with a reduced risk factor space.

[22] $\hat{f}(\boldsymbol{\tau})$ has to be chosen by the management or defined by the regulatory authority. In Switzerland, a lower bound on interest rates has been subject to debate, e.g. FINMA introduced a lower bound on spot rates at −50 basis points for all SST scenarios. See, for example, Swiss Financial Market Supervisory Authority (2012).

For further convenience, we define $\mathbf{a}_t := \hat{f}(\boldsymbol{\tau}) - \mathbf{A} \cdot \boldsymbol{\theta}_t$ and write

$$\Omega_t = \{\boldsymbol{\epsilon}_t | \mathbf{A} \cdot \boldsymbol{\epsilon}_t \geq \boldsymbol{a}_t\}. \tag{20}$$

One should be aware that $\boldsymbol{a}_t$ is a random variable.[23] In this setting the scale and location parameters of the truncated normal distribution are time invariant by assumption but the inequality constraints and hence the integration region are random and vary over time. To demonstrate this more clearly consider the following example.

Let us assume that, at time t, the interest rate is high. In such a case, the inequality constraint $\mathbf{A} \cdot \boldsymbol{\epsilon}_t \geq \boldsymbol{a}_t$ is only weakly binding and of less relevance. Now consider a significant decrease in interest rates from time t to $t+1$, expressed by a change in risk factors $\boldsymbol{\Delta\theta}_{t+1}$. This exogenous shock of interest rates leads to an increase in the lower bound $\mathbf{a}_{t+1}$, $\mathbf{a}_{t+1} = \mathbf{a}_t - \mathbf{A} \cdot \boldsymbol{\Delta\theta}_{t+1}$, which, in consequence, makes the condition $\mathbf{A} \cdot \boldsymbol{\epsilon}_{t+1} \geq \boldsymbol{a}_{t+1}$ much stronger binding at $t+1$ than it was at t. From this example it should be evident that any change in interest rates also changes the domain of $\boldsymbol{\epsilon}_t$.

We conclude this section by stating the main advantages of our approach of using a truncated Gaussian:

1. the aggregation of risk factors becomes straightforward due to dependencies given via a scale matrix;
2. marginal distributions of interest rates become positively skewed and bounded from below.

3.3.1 Properties of the plane-truncated Gaussian distribution

Two properties of this class of plane-truncated Gaussian distributions are vital for the construction of a parameter estimator.

Property 1 *The truncated Gaussian distribution $TN_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma} \mid \Omega_t)$ is proportional to the Gaussian distribution for $\boldsymbol{A} \cdot \boldsymbol{\epsilon}_t \geq \boldsymbol{a}_t$*

$$TN_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma} \mid \Omega_t) = \begin{cases} \frac{1}{\alpha_t} N_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) & \boldsymbol{A} \cdot \boldsymbol{\epsilon}_t \geq \boldsymbol{a}_t \\ 0 & \boldsymbol{A} \cdot \boldsymbol{\epsilon}_t < \boldsymbol{a}_t \end{cases} \tag{21}$$

[23] We allow for time-variant inequality constraints, Ω_t.

The normalisation factor α is determined as the integral of the density over the region being specified by Ω_t,

$$\alpha_t = \int\limits_{\Omega_t} N_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \mathbf{d}\boldsymbol{\epsilon}_t \,. \tag{22}$$

In Property 1, we introduce the following notation: $N_{\boldsymbol{\epsilon}_t}(.,.)$ denotes the multi-dimensional Gaussian density function of the error term (as indicated by the subscript) $\boldsymbol{\epsilon}_t$ with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. It is particularly important to keep in mind that the location and scale parameters of the truncated Gaussian distribution *are not identical* with the expected values and the covariance matrix of the truncated Gaussian.

Property 2 *The conditional expectation value of a truncated Gaussian distribution $TN_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma} \mid \Omega_t)$ is given by*

$$E[\boldsymbol{\epsilon}_t \mid \Omega_t] = \frac{1}{\alpha_t} \int\limits_{\Omega_t} \boldsymbol{\epsilon}_t N_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \mathbf{d}\boldsymbol{\epsilon}_t =: \hat{\boldsymbol{\mu}}_t \,, \tag{23}$$

and the conditional covariance matrix is determined by

$$Var[\boldsymbol{\epsilon}_t \mid \Omega_t] = \frac{1}{\alpha_t} \int\limits_{\Omega_t} (\boldsymbol{\epsilon}_t - E[\boldsymbol{\epsilon}_t]) N_{\boldsymbol{\epsilon}_t}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) (\boldsymbol{\epsilon}_t - E[\boldsymbol{\epsilon}_t])^T \mathbf{d}\boldsymbol{\epsilon}_t =: \hat{\boldsymbol{\Sigma}}_t. \tag{24}$$

3.3.2 Parameter estimator

We already highlighted that, in our setting, $\boldsymbol{a}_t$ is time-dependent. This has two major implications for estimating the time invariant parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Firstly, each sample observation differs in the conditional expected value, and secondly, the data is heteroskedastic. This complicates the parameter estimation. However, by exploiting Property 2, we are able to set up a just identified moment estimator with the following moment conditions.[24]

$$\mathbf{g}_t(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \begin{bmatrix} (\boldsymbol{\epsilon}_t - \hat{\boldsymbol{\mu}}_t) \\ (\boldsymbol{\epsilon}_t - \hat{\boldsymbol{\mu}}_t)^2 - \hat{\boldsymbol{\sigma}}_{i,i,t} \\ (\epsilon_{1,t} - \hat{\mu}_{1,t})(\epsilon_{2,t} - \hat{\mu}_{2,t}) - \hat{\sigma}_{1,2,t} \\ (\epsilon_{1,t} - \hat{\mu}_{1,t})(\epsilon_{3,t} - \hat{\mu}_{3,t}) - \hat{\sigma}_{1,3,t} \\ (\epsilon_{2,t} - \hat{\mu}_{2,t})(\epsilon_{3,t} - \hat{\mu}_{3,t}) - \hat{\sigma}_{2,3,t} \end{bmatrix} \tag{25}$$

[24] $\hat{\sigma}_{i,j,t}$ and $\hat{\mu}_i$ are the specific elements of the covariance matrix $\hat{\boldsymbol{\Sigma}}_t$ and the mean vector $\hat{\boldsymbol{\mu}}_t$ respectively. Moreover $\mathbf{g}_t(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a vector of size nine.

Under the assumption of a random walk for $\boldsymbol{\theta}_t$ and under the null hypothesis of $\boldsymbol{\epsilon}_t$ being a conditionally truncated Gaussian, it follows that $E[\mathbf{g}_t(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \mid \Omega_t] = \mathbf{0}$. The idea behind a Generalised Method of Moments estimator is to replace the orthogonality condition $E[\mathbf{g}_t(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \mid \Omega_t]$ with its sample analogue,

$$\mathbf{m}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{T} \sum_{t=1}^{T} \mathbf{g}_t(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{26}$$

and to choose parameters in such a way that the following quadratic form is minimised:

$$q(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathbf{m}(\boldsymbol{\mu}, \boldsymbol{\Sigma})^T \cdot \boldsymbol{W} \cdot \mathbf{m}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{27}$$

$\mathbf{W}$ is a positive definite matrix that does not depend on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ but may depend on the data. In this application the number of moment conditions matches the number of estimated parameters. Therefore it suffices to find the parameters which solve $\mathbf{m}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathbf{0}$ or minimise the quadratic form $q(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathbf{m}(\boldsymbol{\mu}, \boldsymbol{\Sigma})^T \mathbf{m}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. As a result, in our case the matrix $\mathbf{W}$ defaults to the identity matrix $\mathbf{I}$.

The main hurdle in minimizing Equation (27) lies in finding values for $[\hat{\mu}_{i,t}, \hat{\sigma}_{i,j,t}]$ at each point in time since the cumulative density of a plane-truncated multivariate Gaussian has no analytical expression and has to be evaluated numerically.[25] The most straightforward method for evaluating the integrals in Property 2 is by means of Monte Carlo integration. We implemented a rejection sampler to calculate $\hat{\boldsymbol{\mu}}_t$ and $\hat{\boldsymbol{\Sigma}}_t$ numerically at each point in time. The rejection sampler consists of the following steps:

1. Given the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, one simulates N vectors from the multidimensional Gaussian.

2. Those simulations that do not fulfil the constraint $\mathbf{A} \cdot \boldsymbol{\epsilon}_t \geq \boldsymbol{a}_t$ have to be rejected.

3. From the remaining sample, the expected value and the covariance matrix are calculated. This leads to estimates of $\hat{\boldsymbol{\mu}}_t$ and $\hat{\boldsymbol{\Sigma}}_t$.

4. The procedure is repeated for each point in time.

Given the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, the rejection sampler delivers the conditional expected values and the scale matrix at each point in time. If both the sample size T and the

[25] The moments of the plane-truncated normal distribution were extensively discussed by Tallis (1965). Algorithms for approximating the cumulative density of the plane-truncated normal distribution were examined by Börsch-Supan and Hajivassiliou (1993).

number of simulations N approach infinity, the combination of the rejection sampler with the moment estimator leads to consistent parameter estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

3.3.3 Excursus: The truncated Gaussian process and mean reversion

In this section, we elaborate on an important property of a truncated Gaussian process. For the moment, we shall concentrate solely on an univariate time series process where the innovation ϵ_t is distributed according to a truncated Gaussian. As an example, consider the 10-year to maturity nominal spot rate (r) which is empirically bound from below by a. We furthermore assume that the spot rate's data generating process is given by a non-stationary process, $r_{t+1} = r_t + \epsilon_t$. The innovation ϵ_t follows a conditionally truncated Gaussian, $TN(\mu, \sigma \mid \Omega_t)$.

Conditioning on a lower bound leads to the existence of the area Ω_t and guarantees that the spot rate does not drop below a at any point in time. Hence, the innovation ϵ_t needs to be drawn out of $\Omega_t = \{\epsilon_t | \epsilon_t \geq a - r_t\}$. Assume that today's spot rate is $r_t = r^*$ such that the conditional expectation of ϵ_t equals zero, $E[\epsilon_t \mid \Omega_t] = 0$.[26] Under these assumptions, a negative exogenous interest rate shock decreases r_{t+1}; consequently, the inequality constraint $\epsilon_{t+1} \geq a - r_{t+1}$ becomes more strongly binding and the expectation of ϵ_{t+1} becomes positive $E[\epsilon_{t+1} \mid \Omega_t] > 0$. To phrase it another way, the accessible ϵ-space becomes more and more restricted from the lower bound and is shifted to higher values of ϵ_{t+1} as more and more drawings have to be rejected. Finally, the interest rate is pulled backwards towards r^*.

Conversely, consider that today's interest rate is $r_t = r^*$ and the interest rate undergoes a positive exogenous shock. This will increase r_{t+1} and will relax the constraint $\epsilon_{t+1} \geq a - r_{t+1}$; hence the expectation of ϵ_{t+1} becomes negative, $E[\epsilon_{t+1} \mid \Omega_t] < 0$. Again, the spot rate is pulled towards r^* but now this pull is negative.

In summary, the presented example describes a non-stationary time series process that is mean-reverting and possesses a long-run mean of r^*.[27] We now formulate the following property.

Property 3 *The random process $r_{t+1} = r_t + \epsilon_t$ with $\epsilon_t \sim TN(\mu, \sigma \mid \Omega_t)$ where $\Omega_t = \{\epsilon_t | \epsilon_t \geq a - r_t\}$ and $r_{t=0} \geq a$ is mean-reverting, if and only if, a quantity r^* exists that*

[26] This necessarily implies that μ is negative; otherwise, no r^* with $E[\epsilon_t \mid \Omega_t] = 0$ would exist.
[27] For a detailed proof see the Appendix.

solves

$$\int_{a-r^*}^{\infty} \epsilon_t N_{\epsilon_t}(\mu, \sigma) \mathrm{d}\epsilon_t = 0. \tag{28}$$

The required quantity r^* denotes the long-run mean of the process.

Property 3 highlights an important difference between the process in Equation (16) and the SST standard model. The SST standard model assumes that the risk drivers follow a classical random walk with normal innovations. This leads to symmetric and non mean-reverting interest rate distributions. In sharp contrast, the truncated Gaussian process leads to skewed interest rate distributions in a low interest rate environment; depending on the parameter estimates, the truncated Gaussian process will also drift towards a long-run mean.

3.4 Model recipe

The previous sections outlined a number of theoretical aspects underlying our methodology towards a new SST interest rate risk model. In this section, we briefly summarise the practical target capital sampling recipe on a step-by-step basis.

1. Monthly spot rate data have to be obtained from a data source (e.g. SNB, Deutsche Bundesbank, Bloomberg, etc.).
2. A time series of the relevant risk factors has to be determined. The relevant risk factors are given by the coefficient vector $\boldsymbol{c}_t = [c_{1,t}, c_{2,t}, c_{3,t}]$ of the Nelson and Siegel (1987) Model. The coefficient vector is determined by ordinary least squares (OLS) regression.[28] Spot rates need to be regressed on the factor loadings; this procedure has to be repeated for every month.
3. The parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated by minimising Equation (27).
4. The distribution of the coefficient vector in $t + 12$ (to produce the realisations of interest rates after one year) has to be simulated.
   (a) The simulation is started with $\boldsymbol{\epsilon}_t \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ as the proposal.
   (b) The risk factors of the model, i.e., the coefficient vector $\hat{\boldsymbol{\theta}}_{t+12}^{i=1}$, need to be predicted within the first iteration $i = 1$.

[28] If λ is fixed (e.g. at $\lambda = 0.0609$), Equation (6) is linear in the parameters. Consequently, the parameters can be estimated by OLS.

(c) The truncation constraints on the instantaneous forward rates at $\boldsymbol{\tau} = [0, 12, 24, ..., 600]$ need to be tested; if the instantaneous forward rates are at least higher than the threshold $\hat{f}(\boldsymbol{\tau})$, i.e. $f(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}_{t+12}^{i=1}) \geq \hat{f}(\boldsymbol{\tau})$, the simulated coefficient vector will be accepted.

(d) The number of iterations i has to be increased and steps a) to d) need to be repeated N times in order to obtain a distribution of the Nelson-Siegel coefficients.

5. For your information: The $r(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}_{t+12}^{i})$ can be calculated $\forall\ i$ to obtain the simulated densities of the spot rates.

6. Finally, the sensitivities of Risk Bearing Capital with respect to the Nelson-Siegel coefficients have to be calculated and can be used for approximating the valuation function of the RBC to determine the Target Capital.

4 Data

In this study we use end-of-month government spot rates published by the SNB and the Deutsche Bundesbank. Both central banks use the Svensson (1994) model to extract the zero curve from coupon paying bonds. Hence, the original data generating process of the term structure of interest rates is fully specified by Equation (5). Our sample comprises spot rates from January 2000 to October 2012 with maturities ranging from one to thirty years.

Table 2 presents descriptive statistics of the spot rates. It shows the mean and the first order autocorrelation of spot rate levels. Since the average spot rate increases with time to maturity, the average yield curve is upward sloping. The second column shows the standard deviations of the first differences since in a random-walk case, the standard deviation of interest rate levels is infinite. These decrease with time to maturity. The hypothesis of changes of spot rates being Gaussian is almost always rejected for short and long maturities, as indicated by the p-value of the Jarque-Berra test statistic. Solely medium-term Swiss interest rates might be Gaussian. This stands in contrast to the findings of De Pooter (2007) who examined US treasury zero coupon bonds and rejected the normal distribution solely for medium and long-term maturities.

The observation of first order sample autocorrelations close to 1 for all maturities and currencies indicates a high persistence of shocks and might point towards a non-stationary

	Switzerland				Germany			
Maturity	Mean	SD	ρ_1	JB-p	Mean	SD	ρ_1	JB-p
12 months	1.356	0.181	0.980	0.000	2.487	0.212	0.978	0.000
24 months	1.476	0.179	0.975	0.000	2.676	0.230	0.970	0.000
36 months	1.637	0.180	0.970	0.000	2.875	0.230	0.965	0.429
48 months	1.808	0.172	0.968	0.002	3.068	0.223	0.962	0.691
60 months	1.967	0.165	0.966	0.413	3.249	0.215	0.959	0.463
72 months	2.109	0.160	0.964	0.626	3.414	0.209	0.957	0.378
84 months	2.231	0.158	0.963	0.559	3.562	0.203	0.955	0.365
96 months	2.338	0.157	0.961	0.483	3.695	0.200	0.953	0.358
108 months	2.429	0.156	0.961	0.358	3.812	0.197	0.951	0.316
120 months	2.509	0.156	0.960	0.244	3.916	0.195	0.950	0.234
240 months	2.948	0.148	0.962	0.000	4.437	0.190	0.949	0.000
360 months	3.121	0.145	0.966	0.000	4.524	0.203	0.953	0.000

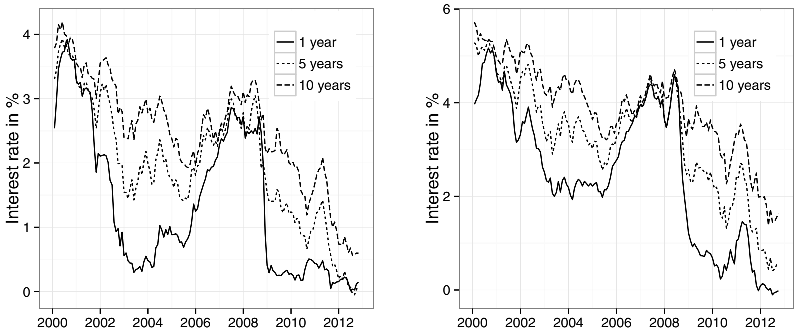
Table 2: Summary statistic for end-of-month yields. The sample period is January 2000 to October 2012 (N=154). Reported are mean as well as first order sample autocorrelation (ρ_1) of interest rates levels. Additionally the standard deviation and the p-value of the Jarque-Berra test statistic for nomality of first differences of end-of-month yields are shown.

time series process. This would imply that the practitioners' intuitively appealing assumption of stationary interest rates might be misleading. On the other hand, the high sample autocorrelation may also be driven by the downward trend in yield curve levels observed in the past decade; this is depicted in Figure 3.

From Figure 3, information about the shape of the yield curve can be inferred as well. For example, the Swiss 1-year spot rate exceeded the 5-year spot rate towards the end of 2008 and in 2012; hence the Swiss yield curve was hump-shaped. The German government curve was almost flat at the beginning of 2008. Moreover, the sharp decrease in the interest rate level towards the end of 2008 is impressive. For example, the Swiss 1-year interest rate dropped from 2.408% (October 2008) to 0.691% (December 2008) and has not reverted to its original level since that date. Finally, Swiss interest rate levels are, on average, lower than the German levels.

5 Results

In this section, we empirically examine the proposed framework for modelling interest rate risk in detail. First, we will assess the in-sample fit of the Nelson-Siegel model. Next, we



(a) Switzerland (b) Germany

Figure 3: Time series plots of a subset of maturities of end-of-month zero coupon yields. The sample period is January 2000 to October 2012 (N=154).

analyse the model's capability to replicate observed interest rate distributions. Finally, we focus on simulating the interest rate distributions for $t + 12$ as the basis for deducing the model's effect on solvency capital requirements. The result is then compared with the SST standard model.

5.1 In-sample fit

A simplification of a spot rate model such as the one set out in Section 3.1.2 will of course lead to a deteriorating model fit. In order to assess the implications arising from application of the parsimonious model we estimate the Nelson-Siegel model coefficients using ordinary least squares and compare the model predictions with the actual yield curve based on Svensson (1994). Table 3 shows the in-sample fit summary statistics of the Nelson-Siegel model estimated over the time period January 2000 to October 2012. The model has been fitted for each month t and the error is calculated as the difference between the model estimate and the yield curve published by the Deutsche Bundesbank or the Swiss National Bank.

Generally speaking, the summary statistics presented in Table 3 show that usage of the Nelson-Siegel model instead of the Svensson model does not harm the model fit. The first three columns present the in-sample fit of the re-engineered Swiss yield curve. A mean absolute error (MAE) below six basis points over all maturities indicates rather satisfying properties of the methodology. For Germany, the MAE is somewhat higher and peaks at maturities around three years. However, an average absolute error of 10.7 basis points still indicates pleasing nature of the applied approximation. This finding is

	Switzerland				Germany			
Maturity	q10	q90	SD	MAE	q10	q90	SD	MAE
12 to 360 months	−9.46	9.98	8.10	5.83	−19.15	15.45	14.14	10.69
12 months	−11.08	1.92	5.18	5.69	−16.31	17.83	13.25	11.94
24 months	−2.31	15.15	7.41	8.48	−11.77	19.20	11.87	11.85
36 months	−3.04	15.07	7.00	7.19	−26.75	22.87	19.24	17.03
48 months	−1.48	7.43	3.66	3.27	−29.70	15.94	18.85	15.44
60 months	−3.02	1.22	1.92	1.58	−29.04	7.45	15.21	11.34
72 months	−7.29	2.07	3.60	4.02	−23.75	0.65	10.51	8.77
84 months	−11.46	2.16	5.17	5.45	−18.54	−2.63	6.00	8.27
96 months	−13.24	2.15	6.10	6.04	−12.83	−2.88	3.78	7.48
108 months	−13.96	1.87	6.51	6.08	−12.69	3.26	5.99	7.29
120 months	−13.37	1.30	6.54	5.76	−14.33	10.32	9.33	8.76
240 months	−2.19	9.26	4.50	4.84	−7.21	39.18	20.28	14.94
360 months	−4.46	24.93	12.34	11.60	0.43	14.33	6.10	5.12

Table 3: In-sample fit: summary statistics in basis points, estimation window from January 2000 to October 2012. The table reports the 1^{st} and 9^{th} decile of the residuals, as well as the standard deviation and the mean absolute error.

in line with De Pooter (2007), Dahlquist and Svensson (1996), and Diebold, Rudebusch, and Boragăn Aruoba (2006), who demonstrate that the parsimonious three-factor Nelson and Siegel (1987) model fits the term structure well compared with more complex term structure models. We therefore conclude that for an adequate risk model it suffices to model $\boldsymbol{\theta}_t = [c_{1,t}, c_{2,t}, c_{3,t}]$ as interest rate risk factors.

Table 4 displays the descriptive statistics of the estimated Nelson-Siegel factors. Both yield curves for Switzerland and Germany behave similarly. The level of Swiss interest rates, represented by $c_{1,t}$, is on average 1.5 percentage points lower than the German level. The German yield curve factors exhibit a somewhat higher standard deviation which follows directly from the higher volatility of German spot rates (see Table 2). The autocorrelation of the coefficients is significant.

In contrast to many other term structure models, e.g. the Svensson model, the Nelson-Siegel model leads to medium-sized cross-correlation between factors as indicated in Table 4. This makes the model particularly interesting for risk management purposes.

In our opinion,

- the satisfying model fit,
- the low cross-correlation between factors, and

- the parsimony in terms of risk factors

make the Nelson-Siegel term structure model a natural choice for modelling interest rate risk.

	Switzerland			**Germany**		
Coefficients	$\hat{c}_{1,t}$	$\hat{c}_{2,t}$	$\hat{c}_{3,t}$	$\hat{c}_{1,t}$	$\hat{c}_{2,t}$	$\hat{c}_{3,t}$
Mean	3.280	−1.414	−3.804	4.750	−1.894	−3.831
SD	0.888	1.113	1.724	1.055	1.569	2.329
ρ_1	0.957	0.938	0.857	0.946	0.962	0.835
JB-p	0.014	0.018	0.920	0.055	0.059	0.031
$\hat{c}_{1,t}$	1.000	–	–	1.000	–	–
$\hat{c}_{2,t}$	−0.227	1.000	–	0.336	1.000	–
$\hat{c}_{3,t}$	−0.110	0.209	1.000	−0.510	−0.280	1.000

Table 4: Estimation results for the Nelson-Siegel coefficients. Rows 1 to 4 show the mean, the standard deviation, the p-value of the Jarque-Berra test statistic for normality and the first sample autocorrelation. Rows 5 to 7 show the correlation matrix of the coefficients.

5.2 Sampling spot rate distributions

With the parsimonious Nelson-Siegel model at hand, we now turn to the question of whether the interest rate distributions resulting from this model in connection with the introduction of plane-truncated error terms can match empirically observed patterns. It is not evident *a priori* that our methodology will be capable of replicating observed interest rate distributions. Indeed, the ability to produce spot rates that *resemble* empirical data is merely a question of the appropriateness of the assumed stochastic process underlying $\boldsymbol{\theta}_t$. Furthermore, this question cannot be answered by means of comparison with the in-sample model fit.

In order to assess the stochastic properties of our methodology, we need to simulate monthly changes in the yield curve – and by that we obviously also replicate changes in interest rates for all maturities. In a second step we compare moments of the changes in the simulated interest rates with the empirically observed moments for each sample path, respectively.

The procedure is straightforward and was essentially stipulated in the model recipe. We first estimate the parameters of the truncated Gaussian $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by GMM over the whole sample. Having obtained the location and scale parameters, we simulate 10,000 paths of spot rate curves starting in January 2000 up to October 2012. From this procedure, we obtain a matrix of predictions of $\hat{\boldsymbol{\theta}}_t$ that all fulfil the given constraints. It is then possible to evaluate the function of the spot rate term structure (Equation (6)) for any given maturity and for each path in order to arrive at the distribution of simulated spot rates. Lastly, we compare the moments of the first differences in spot rates with the moments of the empirically observed spot rate differentials. For this analysis, we used plane-truncated Gaussian innovations with a lower bound of -50 basis points for instantaneous forward rates ($f(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}_{t+1}) \geq -0.50$).

Table 5 compares observed sample moments of the 5- and 10-year to maturity spot rates with the moments of the corresponding simulated data. For each of the 10,000 draws we calculate the moments presented in the columns of the displayed table. Henceforth, the percentiles and the means of these simulated moments are computed i.e. the rows of Table 5. The average Kurtosis of all draws in our Monte Carlo analysis therefore is 2.997 for 5 years to maturity Swiss spot rates. We then may compare this figure to our in-sample estimate of actual Swiss spot rates that is 3.175. This number is well below the third quartile of the simulated Kurtosis (3.207) and we therefore conclude that for this moment our process is able to replicate the data generating process at work.

An in-depth inspection shows that since in January 2000 the interest rate level was fairly high, the constraints were hardly binding at the starting point of the simulations and the innovation was consequently almost Gaussian. This is reflected in the simulated data that show an average skewness close to zero and an average kurtosis of roughly 3. Since none of the sample moments in Table 5 are outside the confidence interval we conclude that the risk model is able to generate empirically observable spot rate distributions.

One should recall that the results in Table 5 arise from linear combinations of $\hat{\boldsymbol{\theta}}_t$, rather than directly from simulations of spot rates. It is also important to stress once more that with the methodology at hand *any* spot rate can be derived. The decision to present the 5-year and 10-year spot rates is somewhat arbitrary; however, it is straightforward to carry out this analysis for further maturities with equivalent results. As indicated in Table 5 it suffices to model $\boldsymbol{\theta}_t$ to obtain accurate yield distributions – without simulating yields directly. The analysis highlights the flexibility of the proposed approach by showing that it is able to reproduce empirically observed data generating processes for interest rates of any maturity.

	CH 60 months				CH 120 months			
	Mean	SD	Skewness	Kurtosis	Mean	SD	Skewness	Kurtosis
q1	−0.024	0.139	−0.474	2.330	−0.028	0.123	−0.475	2.313
q25	−0.019	0.155	−0.137	2.729	−0.022	0.136	−0.136	2.707
mean	−0.014	0.161	−0.008	2.997	−0.016	0.142	−0.006	2.974
q75	−0.011	0.168	0.121	3.207	−0.011	0.147	0.123	3.175
q99	0.007	0.184	0.461	4.206	0.006	0.161	0.467	4.164
Sample Est.	−0.021	0.165	−0.248	3.175	−0.021	0.156	−0.320	3.182
	DE 60 months				**DE 120 months**			
	Mean	SD	Skewness	Kurtosis	Mean	SD	Skewness	Kurtosis
q1	−0.034	0.189	−0.472	2.310	−0.039	0.161	−0.472	2.309
q25	−0.026	0.211	−0.135	2.725	−0.030	0.179	−0.130	2.702
mean	−0.019	0.220	−0.006	2.992	−0.021	0.186	−0.003	2.969
q75	−0.014	0.229	0.124	3.201	−0.013	0.193	0.128	3.173
q99	0.011	0.251	0.456	4.150	0.010	0.212	0.463	4.149
Sample Est.	−0.031	0.215	−0.141	2.596	−0.027	0.195	−0.334	3.110

Table 5: Comparison of simulated and empirical moments of first differences in spot rates. Rows show the mean, the 1^{st}, 25^{th}, 75^{th} and 99^{th} percentile of the simulated moments.

5.3 Comparison with the SST standard model

In the preceding section, we showed that the combination of the Nelson-Siegel model with plane-truncated Gaussian innovations leads to a risk model that is able to replicate empirically observed moments of spot rate distributions. Nevertheless, it has to be admitted that the results in Table 5 could also have been produced using just Gaussian innovations; *from this point of view* we failed to provide motivation for the usage of a plane truncation. In this section, we address this issue.

The appropriateness of the assumption of truncated innovations is motivated by the firm conviction that a lower bound on nominal interest rates exists. For the analysis executed in this section we applied a plane truncation to the innovations such that the instantaneous forward rates at $\boldsymbol{\tau} = [0, 12, 24, ..., 600]$ were higher than -50 basis points. The introduction of a lower bound to Gaussian innovations will alter the shape of the spot rate distributions significantly. In Subplot (a) of Figure 4, we depict twelve-months-ahead predictions of the 1-year to maturity spot rate. The starting point of the prediction is the observed spot rate as at October 2012. In order to compare our model forecasts with those of the SST standard model, we superimpose the density distribution of the SST standard model as a grey line.

In October 2012 the spot rate level was very low, thus the truncation is highly effective. As a consequence, a large chunk of density mass is moved from the negative towards the positive; the resulting distribution resembles a shifted Log-Normal distribution. Given the truncation constraint, one might have expected a sharp cut of the density distribution at -50 basis points; however, the figure proves this assumption wrong. The resulting density function is rather smooth. The smoothness of the spot rate density has its roots in one distinctive feature of the truncation. Actually, the truncation of the space of $\boldsymbol{\theta}_t$ acts as a constraint on the instantaneous forward rates rather than directly on the spot rates. As a result, the simulated spot rate density distributions become smooth.

Figure 4 gives a graphical impression of the model mechanics. Subplots (a) and (b) depict interest rates of the two maturities 1 and 10 years. Subplots (b) and (c) show the implications from different interest rate regimes. The only difference between (b) and (c) is the starting point of the simulation. In Subplot (b), the starting yield is the spot rate as at October 2012 (0.6%), while in Subplot (c) the interest rate as at January 2000 (3.8%) was used. When we compare our model with the SST standard model in Subplot (b), we find that the expectation of the spot rate in $t + 12$ implied by our model is shifted to the right. Even more interestingly, the model implied expectation in Subplot (c) is shifted to the left when compared with a pure Gaussian without truncation. This graphically highlights the point made in Section 3.3.3: We argue that, depending on the parameter estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, a non-stationary truncated Gaussian process leads to mean-reverting spot rates. As it happens, the parameter estimator suggests that spot rates are actually mean-reverting.

The economic implications of the introduction of truncated Gaussians are of fundamental importance for insurers. As set out in Section 1, the SST standard model sets interest rate distributions disregarding the current state of the economy. This results in target capital requirements that are independent of the state of the economy. The model we suggest adapts to the state of the economy. Compared to our findings, the SST standard model results in lower solvency capital requirements during high yield phases where downward potential is high. At the same time, the standard model leads to relatively higher capital requirements in low yield regimes when interest rate downward potential approaches zero due to the lower bound on interest rates. From a macroeconomic perspective, the SST standard model may cause insurers to potentially over-invest in boom phases and hold back funds during bust environments. This pattern is known as procyclicality.[29] The proposed model, however, introduces a countercyclical momentum since capital requirements correspond to the prevailing interest rate environment. Firstly, interest rate

[29] See, for example, Blum and Hellwig (1995) for an early discussion on this issue in the banking sector.

distributions are skewed in a low interest rate environment and rather symmetric when the prevailing interest rate level is high. Secondly, the centre of the distribution converges towards its long-run mean. Comparison of Subplot (b) and Subplot (c) highlights the countercyclicality of the model.

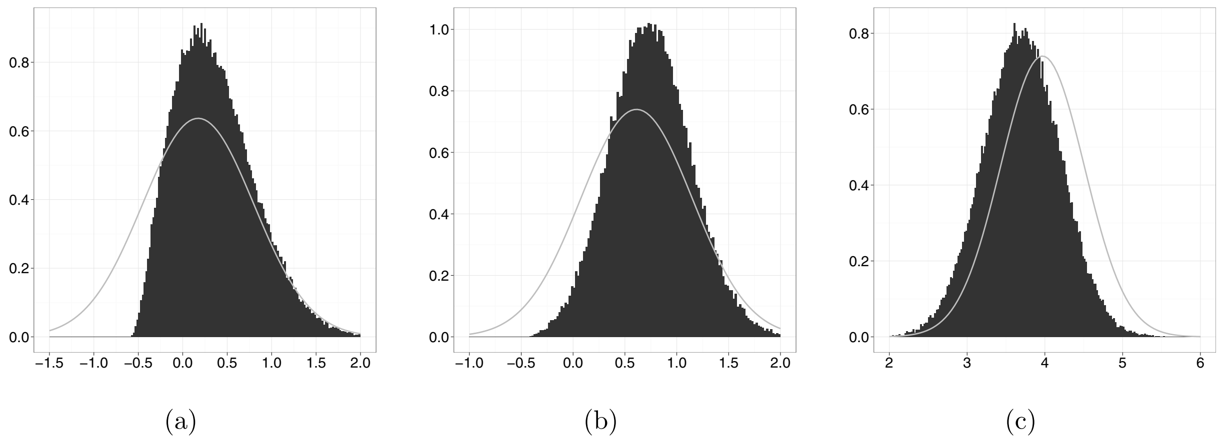


Figure 4: Comparison of spot rate density distributions. The black area illustrates the simulated density of the discussed model. The grey line shows the yield density distribution of the SST standard model. Subplots (a) and (b) depict the interest rates distributions for 1 and 10 years to maturity for $t + 12$ as at reporting date October 2012. Subplot (c) shows the 10 year to maturity interest rate distribution for $t + 12$ as at January 2000.

6 Conclusion

The Swiss Solvency Test requires insurers to develop a market-consistent valuation of assets and liabilities. The calculation of solvency capital requirements builds on market-consistent valuation as well. Since market values of insurance liabilities are very thin on the ground, interest rates are key for the valuation of liabilities and for the calculation of capital requirements from asset-liability mismatch.

This paper analyses the treatment of interest rate risk under the SST with the standard model and finds three essential shortcomings. Firstly, the standard risk model suggests a considerable number of interest rate risk factors. The risk manager is confronted with the complex and laborious task of modelling and numerically simulating thirteen interest rate buckets – for each currency. Such a requirement becomes even more futile when the data generating process of spot rate yield curves is considered. Secondly, under the SST, changes in spot rates are assumed to be Gaussian. This allows for highly negative interest rates – especially in low interest rate environments such as those faced during the ongoing financial and economic crisis. Thirdly, and most importantly, in the case of a positive interest rate sensitivity of the Risk Bearing Capital, the utilization of Gaussians automatically introduces procyclical capital requirements. The standard model forces the insurance undertaking to hold more solvency capital in a low interest rate environment even though the downward interest rate risks is less pronounced than usual.

The new methodology concentrates on modelling interest rates and addresses the flaws of the SST. Firstly, a systematic analysis of the term structure of spot rates suggests a reasonable simplification of the risk model. By explicitly considering of the data generating process for spot rate curves, the number of risk factors can be significantly reduced to the three parameters of the Nelson-Siegel class of interest rate models. Secondly, the stochastic process used to model spot rates is built around a truncated Gaussian which allows for the introduction of a spot rate floor. The existence of a floor is reasonable from a theoretical point of view since negative interest rates should at most reflect the cost of holding cash. This is also suggested from an empirical analysis as highly negative interest rates have never been observed. Using a Method of Moments-type estimator, the truncated Gaussian process results in a mean-reverting interest rate process that matches empirical observations. Thirdly, the suggested model yields a distribution of interest rates that adapts to the prevailing economic regime. In low interest rate environments, the distribution is positively skewed while in high interest rate environments the spot rate distribution is symmetric. Finally, the paper puts forward a model recipe for implementation and delivers an empirical discussion of the results obtained.

Appendix

The mean-reverting behaviour of a truncated Gaussian process

In order to show that the process

$$r_{t+1} = r_t + \epsilon_t, \qquad \epsilon_t \sim TN(\mu, \sigma, \mid \Omega_t) \qquad \text{where} \qquad \Omega_t = \{\epsilon_t | \epsilon_t \geq a - r_t = a_t\} \tag{29}$$

is converging towards its long-run mean r^*, we have to show that

1. the long-run mean r^* exists and
2. the random number r_t is pulled towards r^*.[30]

[30] It should be noted that at $r_t = r^*$ the conditional expectation of ϵ_t is $E[\epsilon_t \mid \Omega_t] = 0$.

We start exploring the restrictions on μ and σ such that r^* exists. Furthermore, the conditional expectation of the truncated Gaussian distribution is determined by

$$E[\epsilon_t \mid \Omega_t] = \frac{1}{\alpha_t} \int_{a_t}^{\infty} \epsilon_t N_{\epsilon_t}(\mu, \sigma) \mathrm{d}\epsilon_t = \mu + \sigma \cdot \lambda(\beta_t),$$

where $\lambda(\beta_t)$ is the inverse Mills ratio,

$$\lambda(\beta_t) = \frac{\phi(\beta_t)}{1 - \Phi(\beta_t)}.$$

ϕ and Φ denote the standard normal density and cumulative density distribution respectively; $\beta_t = \frac{a_t - \mu}{\sigma}$.[31] As several authors point out, the inverse Mills ratio is monotonically increasing and continuous in β_t, consequently $E[\epsilon_t \mid \Omega_t]$ is continuous and monotonically increasing in β_t and a_t (see Hayashi, 2000, for further reference).

Our strategy is to calculate $E[\epsilon_t \mid \Omega_t]$ at the lower and upper bound of a_t. If $E[\epsilon_t \mid \Omega_t]$ is negative at the lower bound and positive at the upper bound the existence of $E[\epsilon_t \mid \Omega_t] = 0$ follows from continuity of $E[\epsilon_t \mid \Omega_t]$ in a_t. The lower bound of a_t is $a_t = -\infty$ and the upper bound is given by $a_t = 0$ – in this case today's observation is exactly at the bound $a = r_t$. Note that

$$\lim_{a_t \to -\infty} E[\epsilon_t \mid \Omega_t] = \mu.$$

Since $E[\epsilon_t \mid \Omega_t]$ has to be negative at the lower bound it follows that μ has to be negative. $E[\epsilon_t \mid \Omega_t]$ at the upper bound is given by

$$E[\epsilon_t \mid \Omega_t] = \mu + \sigma \cdot \lambda \left(\frac{0 - \mu}{\sigma} \right).$$

Thus, r^* exists, if and only if

$$\mu \leq 0 \leq \mu + \sigma \cdot \lambda \left(\frac{0 - \mu}{\sigma} \right). \tag{30}$$

Given as set μ and σ that fulfil Equation (30) we finally have to show that r_t is pulled to r^*. This is achieved by demonstrating that the expectation of ϵ_t is positive if today's level of r_t is below r^* and negative if r_t is above r^*. We start by showing that if $r_t < r^*$, $E[\epsilon_t \mid \Omega_t] > 0$.

$$a^* = a - r^* < a - r_t = a^{**} \tag{31}$$

31 Again, we condition on the information set at time t. It immediately follows that β_t and a_t are given. Furthermore, note that $N_{\beta_t}(0,1) = \phi(\beta_t)$, whereas $\Phi(\beta_t) = \int_{-\infty}^{\beta_t} N_x(0,1)dx$.

Since $E[\epsilon_t \mid \Omega_t]$ is monotonically increasing in a_t and $E[\epsilon_t \mid \Omega_t] = 0$ at a^*, it follows that $E[\epsilon_t \mid \Omega_t] > 0$ at a^{**}. The converse holds true for $r_t > r^*$ establishing that the truncated Gaussian process defined in Equation (29) is mean-reverting if r^* exists.

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