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Credit risk stress testing and copulas – is the Gaussian copula better than its reputation?

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Non-technical summary

Research Question

In the last decade, stress tests have become indispensable in bank risk management which has led to significantly increased requirements for stress tests for banks and regulators. Although the complexity of stress testing frameworks has been enhanced considerably over the course of the last few years, the majority of credit risk models (e.g. CreditMetrics) still rely on Gaussian copulas which have been strongly criticized by financial experts in the aftermath of the 2008-2009 financial crisis (e.g. Jones, 2009; Salmon, 2009). We challenge this view by investigating the influence of different copula functions in credit risk stress testing.

Contribution

This paper complements the finance literature providing new insights into the impact of different copulas in stress test applications using supervisory data of 17 large German banks. Our comprehensive simulation study allows us to disentangle the main drivers for the observed effects and to explain which copula determines which stress level subject to the chosen input parameters. Furthermore, this paper provides guidance for practitioners, such as risk managers and regulators, on how to design a credit risk stress test and recommends always investigating a variety of dependence structures to determine which specification leads to the adequate stress forecasts.

Results

Our findings imply that the use of a Gaussian copula in credit risk stress testing should not by default be dismissed in favor of a heavy-tailed copula as it is widely recommended in the finance literature. While there might be pitfalls of Gaussian modeling in risk management applications under normal scenarios, one should always be aware of possible counterintuitive effects when truncating distributions as is the case in many stress test approaches.

Nichttechnische Zusammenfassung

Fragestellung

In den vergangenen Jahren sind Stresstests ein unverzichtbarer Teil des Risikomanagements von Banken geworden, was zu deutlich höheren Anforderungen an Stresstests sowohl für Banken als auch für Regulierungsbehörden geführt hat. Wenngleich die Komplexität der Stresstests in den letzten Jahren erheblich gestiegen ist, basiert die Mehrheit der Kreditrisikomodelle (z.B. CreditMetrics) immer noch auf Gauss-Copulas, obwohl diese in Folge der Finanzkrise 2008-2009 stark kritisiert wurden (e.g. Jones, 2009; Salmon, 2009). Wir stellen diese Kritik in Frage, indem wir die Einflüsse verschiedener Copulas in Kreditrisiko-Stresstests untersuchen.

Beitrag

Dieses Papier liefert neue Erkenntnisse über die Auswirkungen verschiedener Copulas in Stresstests anhand bankenaufsichtlicher Daten von 17 deutschen Großbanken. Unsere umfassende Simulationsstudie ermöglicht es, die einzelnen Einflussfaktoren beobachteter Effekte eindeutig zu identifizieren und zu erklären, welche Copula welches Stressniveau unter den gewählten Eingangsparametern bestimmt. Außerdem gibt unsere Studie Risikomanagern wie Regulierern Richtlinien für den Aufbau von Stresstests und gibt die Empfehlung, stets eine Vielzahl von Abhängigkeitsstrukturen zu untersuchen, um die Spezifikation des Stresstests zu wählen, die zu den adäquaten Stressprognosen führt.

Ergebnisse

Unsere Erkenntnisse zeigen, dass, anders als häufig in der Finanzliteratur empfohlen, der Gebrauch der Gauss-Copula in Kreditrisiko-Stresstests nicht grundsätzlich zu Gunsten von heavy-tail Copulas verworfen werden sollte. Auch wenn die Modellierung mit Normalverteilungsannahmen im Risikomanagement unter gewöhnlichen Bedingungen diverse Probleme aufweist, sollte man sich der möglichen kontraintuitiven Effekte durch die Trunkierung von Verteilungen, die in vielen Stresstest-Ansätzen üblich ist, bewusst sein.

Credit risk stress testing and copulas - is the Gaussian copula better than its reputation?*

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Abstract

In the last decade, stress tests have become indispensable in bank risk management which has led to significantly increased requirements for stress tests for banks and regulators. Although the complexity of stress testing frameworks has been enhanced considerably over the course of the last few years, the majority of credit risk models (e.g. Merton (1974), CreditMetrics, KMV) still rely on Gaussian copulas. This paper complements the finance literature providing new insights into the impact of different copulas in stress test applications using supervisory data of 17 large German banks. Our findings imply that the use of a Gaussian copula in credit risk stress testing should not by default be dismissed in favor of a heavy-tailed copula which is widely recommended in the finance literature. Gaussian copula would be the appropriate choice for estimating high stress effects under extreme scenarios. Heavy-tailed copulas like the Clayton or the t copula are recommended in the case of less severe scenarios. Furthermore, the paper provides clear advice for designing a credit risk stress test.

Keywords: credit risk, top-down stress tests, copulas, macroeconomic scenario
JEL classification: G21, G33, C13, C15

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1 Introduction

In the last decade, stress tests have become indispensable in bank risk management. Nowadays, stress tests are a key instrument for risk analysis and banking supervision (e.g. Brunnermeier, Crockett, Goodhart, Persaud, and Shin, 2009; de Larosière, Balcerowicz, Issing, Masera, McCarthy, Nyberg, Pérez, and Ruding, 2009; Turner, 2009). Before the European Central Bank (ECB) assumed banking supervision tasks in November 2014 in its role within the Single Supervisory Mechanism (SSM), the ECB conducted a comprehensive euro-area-wide stress test of the new significant institutions in order to build confidence by assuring all stakeholders that, on completion of the identified remedial actions, banks would be soundly capitalized (European Central Bank, 2014a). Since 2011, the Federal Reserve has been conducting the Comprehensive Capital Analysis and Review (CCAR) (Federal Reserve, 2014a) and the Dodd-Frank Act Stress Test (DFAST) (Federal Reserve, 2014b) on an annual basis to assess the resilience of the largest bank holding companies operating in the US under different scenarios. In the new Supervisory Review and Evaluation Process (SREP) (European Central Bank, 2014b) applied by the SSM, stress tests are a central element, for instance, for assessing institutions' exposures and resilience to adverse but plausible future events. As a matter of course, stress tests play an important role in risk management of individual banks as well (e.g. Basel Committee on Banking Supervision, 2009). Furthermore, CEBS' guidelines on stress testing (Committee of European Banking Supervision, 2010) require banks to consider severe economic downturns under Pillar II capital requirements.

Stress testing frameworks have been developed considerably further over the last few years. In the first years, approaches were characterized by mostly single shocks, limited focus on selected products or business units, static frameworks, no usual link to capital adequacy and one dimensionality solely considering losses. Today, broad macro scenarios and market stress, comprehensive, firm-wide, dynamic and path-dependent, explicit post-stress common equity thresholds, simultaneously losses, revenues and costs are taken into account. This means that stress tests now include many aspects reaching a significant level of complexity (e.g. Borio, Drehmann, and Tsatsaronis, 2014; Schuermann, 2014)[1]. With the increasing importance and heightened uncertainty in financial markets, severity of stress scenarios had to increase as well. Against this background time horizons were also extended significantly which led to an additional increase in the stress effect.

[1]Stress test frameworks for interbank network are even more complex (e.g. Amini, Cont, and Minca, 2012)

Although both the complexity of stress testing frameworks and the severity of adverse scenarios has increased considerably over the course of the last few years, the majority of credit risk models (e.g. Merton (1974), CreditMetrics, KMV) still rely on Gaussian copulas. In the aftermath of the 2008-2009 financial crisis, there has been a strong criticism of mathematics and the mathematical models used by the finance industry, especially the reliance on Gaussian copulas. Jones (2009) and Salmon (2009) thoroughly questioned the usage of the Gaussian copula and tried to explain the limitations of this approach as well as its dangerous role in the 2007-2008 financial crisis. Of course, the drawbacks of light-tailed distributions are not new in the finance literature, as described in detail for instance in Borio, Drehmann, and Tsatsaronis (2010). In general, Genest, Gendron, and Bourdeau-Brien (2009) document the advent and spectacular growth of copula theory. However, the appropriate usage of copulas in finance applications is still far from being clear.

In general, the finance literature very rarely identifies the Gaussian copula as the most appropriate copula for specific applications. Crook and Moreira (2011) apply copula methods to model dependence across default rates in a credit card portfolio of one large UK bank, but they do not stress the credit card portfolio. Their empirical results show that copula families other than the Gaussian one are able to better model the dependence structure of the credit portfolios. The paper by Brechmann, Czado, and Paterlini (2014) reveals that Gaussian and t copulas can provide a good fit to model operational risk. Fischer, Koeck, Schlueter, and Weigert (2009) find that, empirically, the Student t copula outperforms more general Archimedean copulas in terms of goodness of fit measures. However, they also find that the relative performance of the Gaussian copula improves as the number of dimensions increases. According to Diks, Panchenko, and van Dijk (2010) the Student t copula outperforms other specifications in out-of-sample density forecasts when using the Kullback-Leibler information criterion as means of comparison. Hamerle and Roesch (2005) show that a Gaussian copula tends to overestimate the default correlations, as compared to a t copula, implying that in the context of model misspecification, the Gaussian copula might constitute a more conservative approach. The choice of copula (normal versus Student t), which determines the level of tail dependence, has a rather modest effect on risk (e.g. Rosenberg and Schuermann, 2006). For a portfolio consisting of stocks, bonds and real estate, Kole, Koedijk, and Verbeek (2007) provide clear evidence in favor of the Student's t copula and reject Gaussian copula and the extreme value-based Gumbel copula. Junker, Szimayer, and Wagner (2006) analyse the dependence in the term structure of US Treasury yields. They show that the transformed Frank copula has the best overall fit. Hakwa, Jäger-Ambrozewicz, and Rüdiger (2015) propose a flexible framework for the computation of the CoVaR in a very general stochastic setting based on copula theory. When applying both elliptical and Archimedean copulas, the study does not identify one of the copulas as the most adequate one. The study by Kalkbrener and Packham (2015b) is the closest to ours and shows that Gaussian and t copulas behave differently under stress using illustrative examples.[2] In a theoretical study, Kalkbrener and Packham (2015a) investigate correlations of asset returns in stress scenarios and find that correlations in heavy-tailed normal variance mixture models react less sensitively to stress than medium or light-tailed models. However, Choros-Tomczyk, Haerdle, and Overbeck (2014) revisit the analysis of CDO prices and find that an inverse Gaussian copula is superior to other specifications. To sum all these findings up, the choice of the right copula clearly depends on the object under investigation and the degree to which extreme scenarios are modeled. The usage of copulas in stress test applications has not been tackled in detail so far except in Kalkbrener and Packham (2015b).

This study complements the finance literature providing new insights into the impact of different dependence structures in stress test applications. We apply a standard multi-factor credit risk model - CreditMetrics - with sector-dependent unobservable risk factors as drivers of the systematic risk (e.g. Bonti, Kalkbrener, Lotz, and Stahl, 2006; Duellmann and Kick, 2014) and add further copula functions to this framework - both elliptical and Archimedean copulas - in order to achieve more insights into the choice of copula behavior in stress tests. In the first part of the paper, we explore supervisory data of 17 large German banks and measure the impact of the selected copulas on the banks' regulatory capital ratios. For this purpose, highly granular credit risk information on loan volumes and banks' internal estimates of default probabilities are considered in a departure from the majority of stress test studies to cover appropriately the risk concentrations in the banks' credit portfolios. Furthermore, the applied macroeconomic scenario is, on the one hand, parsimonious as well as very intuitive ("financial crisis"-type) and is derived from historical distributions of German GDP per business sectors; on the other hand, it is severe in line with the current trend of more severe scenarios and more complex stress test frameworks (Busch, Koziol, and Mitrovic, 2015). In the second part, a comprehensive simulation study allows us to disentangle the main drivers for the impact of the different copulas in credit risk stress testing and to explain which copula determines which stress level with respect to the chosen input parameters.

In a stress test framework, the key drivers, such as severity of stress effect on each business sector and the correlation between business sectors, are exogenously determined by the

[2] In a similar study, Packham, Kalkbrener, and Overbeck (2016) investigate in particular probabilities of default and default correlations under stress.

macroeconomic scenario which limits the degrees of freedom in executing stress tests. Thus, it is key to understand which copula fits best to the chosen macroeconomic scenario. Against this background, this paper provides guidance for practitioners, such as risk managers and regulators, on how to design a credit risk stress test and shows best practices in using copula functions in stress testing.

Our findings imply that the use of a Gaussian copula in credit risk stress testing should not by default be dismissed in favor of a copula with higher tail dependence. It is important to investigate a variety of dependence structures and determine which specification leads to the appropriate stress forecast. Our comprehensive stress test on 17 German banks reveals that the Gaussian copula produces more severe reductions of the banks' capital ratios than the other heavy-tailed copulas. Even though the differences that appear in terms of basis point capital ratio changes are not large, transforming them to concrete capital positions, these differences are classified as material for banks and regulators. The Gaussian copula would be an appropriate choice for estimating high stress effects in situations if the applied stress scenario is very severe, meaning that it is characterized by extreme cutoff values for a number of business sectors and high sector correlation values possibly combined with a homogenous stress distribution across the affected business sectors. Heavy-tailed copulas like the Clayton or the t copula are recommended in the case of less severe adverse scenarios. Assuming very low correlation values means the t copula generates comparably high stress levels for weak stress scenarios. Clayton copulas are preferable under semi-strong adverse scenarios in which only a limited number of business sectors are directly stressed.

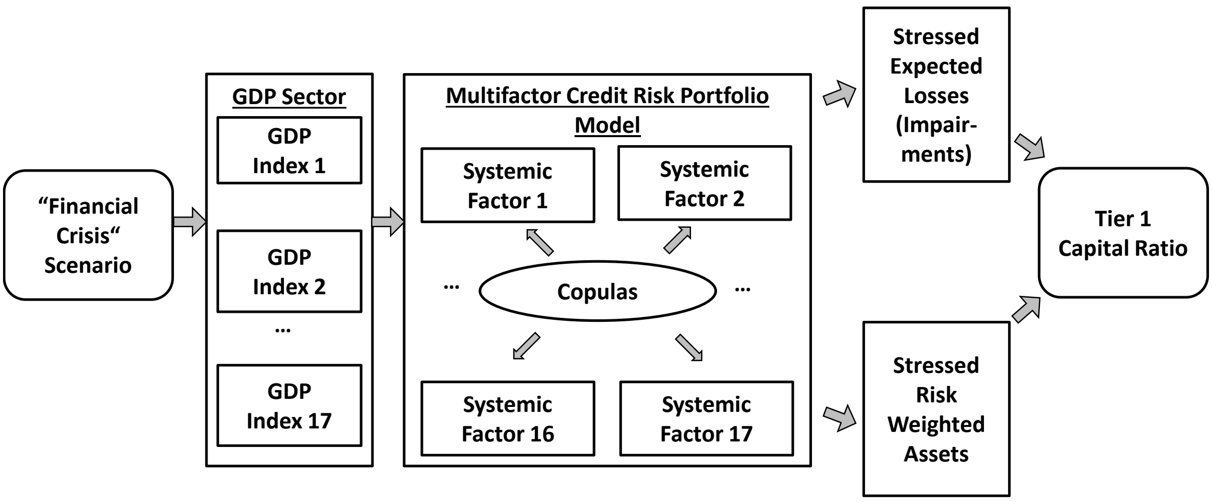
This paper is structured as follows: Section 2 describes the stress test design applied in this study introducing copulas, the credit risk model, the macroeconomic scenario and the supervisory data set. In Section 3 the results of our bank stress tests are presented using different copulas. These, at first glance counterintuitive results, are analyzed within an in-depth simulation study in Section 4 which leads to practical implications for credit risk stress testing in terms of the choice of copulas in Section 5.

2 Stress test design

In this section, we introduce the features of the stress testing approach applied in this study. First, we review some properties of copula functions that are necessary for the modeling of dependence structures in our stress test. Then, we describe the actual stress testing framework that we employ in more detail. The description is separated into an explanation of the credit risk model, the specification of the macroeconomic stress scenario and a summary of the data and the portfolio stress measures that we compute. A broad overview of the stress test design can be found in Figure 1.

Figure 1: Overview of the stress test design

This diagram shows a schematic representation of the stress test design. The individual modules represented as parts of the figure are described in detail in this section.



2.1 Copulas

When looking at a multivariate random vector $X = (X_1, ..., X_n)^T$ with distribution function F, i.e. $F(x_1, ..., x_n) = \mathbb{P}(X_1 \leq x_1, ..., X_n \leq x_n)$, notice that F contains all the information about the margins as well as the dependence structure between the components of X. The mathematical concept of copulas allows us to examine both parts separately and to model dependencies in non-linear contexts adequately.[3]

[3] For a detailed description of copulas, see Embrechts, Lindskog, and McNeil (2003), Cherubini, Luciano, and Vecchiato (2004) or Nelsen (2006).

For the purpose of this paper, all distribution functions and densities are assumed to be continuous.

Definition 1 (Copula) *A copula C is a multivariate distribution function on the n-dimensional unit cube with uniformly distributed marginals on* $[0, 1]$.

To link the idea of copulas to any desired distribution function, we use the standard result of transformations of random variables: if X is a random variable with distribution function F and U is standard uniformly distributed, it holds that $F^{-1}(U) \sim \mathcal{F}$ and $F(X) \sim \mathcal{U}(0, 1)$. The first statement delivers a simple method to sample from the distribution F in first simulating a standard uniformly distributed variable $U \sim \mathcal{U}(0, 1)$ and then setting $X = F^{-1}(U) \sim \mathcal{F}$. $F(X) \sim \mathcal{U}(0, 1)$ assures that every random variable can be transformed into a uniformly distributed random variable on $[0, 1]$ in plugging it into its own distribution function. The following equation now motivates Sklar's theorem linking the multivariate distribution function to its margins and the copula function representing the dependence structure:

$$F(x_1, ..., x_n) = \mathbb{P}(X_1 \leq x_1, ..., X_n \leq x_n) = \mathbb{P}[F_1(X_1) \leq F_1(x_1), ..., F_n(X_n) \leq F_n(x_n)]$$

with $F_i(X_i) \sim \mathcal{U}(0, 1)$.

Theorem 2 (Sklar's theorem) *If F is a multivariate distribution function with univariate marginals $F_1, ..., F_n$, then F can be written as*

$$F(x_1, ..., x_n) = C[F_1(x_1), ..., F_n(x_n)] \qquad \forall x \in \overline{\mathbb{R}}^n$$

for some copula C. In the case of F being continous, C is unique. Conversely, one can define any multivariate distribution function F with univariate marginals $F_1, ..., F_n$ by selecting an arbitrary copula function and setting $F(x_1, ..., x_n) = C[F_1(x_1), ..., F_n(x_n)]$ $\forall x \in \overline{\mathbb{R}}^n$.[4]

In our stress test setup, we take advantage of the second part of Sklar's theorem as we fix the standard normal marginals and choose different copulas for the dependence structure between the systematic risk factors. This method generates different multivariate distribution functions in setting $F = C[F_1, ..., F_n]$ and is therefore called *copula engineering*.

[4]See Nelsen (2006) for the proof.

There is a host of bivariate copulas that can be found in the literature, but that cannot be generalized to higher dimensions. As our study works with a multivariate risk vector, we now take a closer look at those copulas that can be used in higher dimensional applications and that are frequently used in finance applications.

Definition 3 (Classifications) *Let $u = (u_1, ..., u_n)^T \in [0,1]^n$. Then the following copula functions can be defined:*

1. *The Gaussian copula function is given by*

$$C_\Sigma^{Ga}(u) = \Phi_\Sigma\left[\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)\right]$$
$$= \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} \exp\left[-\frac{1}{2}x^T\Sigma^{-1}x\right] dx_1 \cdots dx_n,$$

where $x \in \mathbb{R}^n$, with $\Phi_\Sigma(\cdot)$ being the distribution function of the n-dimensional normal distribution with linear correlation matrix Σ and $\Phi^{-1}(\cdot)$ being the inverse of the univariate standard normal distribution.
C_Σ^{Ga} is the implicit copula function of a multivariate normal distribution, i.e. the copula that "couples" n univariate normally distributed marginals to an n-dimensional normal distribution with correlation matrix Σ. The density of the bivariate normal copula can be written as

$$\lambda_\rho^{Ga}(u_1, u_2) = \frac{1}{\sqrt{(1-\rho^2)}} \exp\left(\frac{2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2) - \rho^2(\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2)}{2(1-\rho^2)}\right)$$

2. *The Student t copula with m degrees of freedom (or t_m copula) is given by*

$$C_{m,\Sigma}^t(u) = t_{m,\Sigma}\left(t_m^{-1}(u_1), ..., t_m^{-1}(u_n)\right),$$

where $t_{m,\Sigma}(\cdot)$ is the implicit copula function of the multivariate t distribution with m degrees of freedom, linear correlation matrix Σ and $t_m^{-1}(\cdot)$ being the inverse of the univariate t-distribution with m degrees of freedom.
The density of the bivariate t_m copula can be written as

$$\lambda_{m,\rho}^t(u_1, u_2) = \frac{\Gamma\left(\frac{m+2}{2}\right)\Gamma\left(\frac{m}{2}\right)\left(1 + \frac{t_m^{-1}(u_1)^2 + t_m^{-1}(u_2)^2 - 2\rho t_m^{-1}(u_1)t_m^{-1}(u_2)}{m\sqrt{(1-\rho^2)}}\right)^{-\frac{m+2}{2}}}{\sqrt{(1-\rho^2)}\Gamma\left(\frac{m+1}{2}\right)^2 \prod_{i=1}^2 \left(1 + \frac{t_m^{-1}(u_i)^2}{2}\right)^{-\frac{m+1}{2}}}$$

3. The Clayton copula with parameter α is given by

$$C(u_1, ..., u_n) = \left[\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right]^{-\frac{1}{\alpha}} \quad \textit{with } \alpha > 0$$

The density of the bivariate Clayton copula can be written as

$$\lambda_\alpha^{Clayton}(u_1, u_2) = (\alpha + 1)(u_1 u_2)^{-(\alpha+1)} \left(u_1^{-\alpha} + u_1^{-\alpha} - 1\right)^{-\frac{2\alpha+1}{\alpha}}$$

The Gaussian and the t_m copula are based on elliptical distribution functions and therefore also called elliptical copula functions. Both can be characterized through the correlation matrix and the degrees of freedom since the Gaussian copula is just a special case of the t_m copula for the degrees of freedom converting to infinity.
The Clayton copula is part of the Archimedean family containing copulas that can be constructed via so-called generator functions φ that have to fulfill certain conditions. A very important advantage of an Archimedean copula is that it can model asymmetric asymptotic dependencies in the tails of a distribution.

Our study is based on the three copula functions Gaussian, t_2 and Clayton where the Gaussian choice is considered the standard that has to be challenged with distribution functions capturing tail dependence. The t copula is a natural extension of the Gaussian one and also frequently used in practice, the Clayton copula out of the Archimedean family is taken for its effects of lower tail dependence.

For the copulas to be still comparable and to show the effect that lies only in the choice of the copula function, parameters are calibrated such that (average) linear correlations as well as marginals etc. are kept fix throughout our study. To be more precise, we first relate the average linear correlation of the Gaussian and t copula approach to an average Kendall's τ and then translate Kendall's τ as a global measure of dependence when determining α for the Clayton copula.

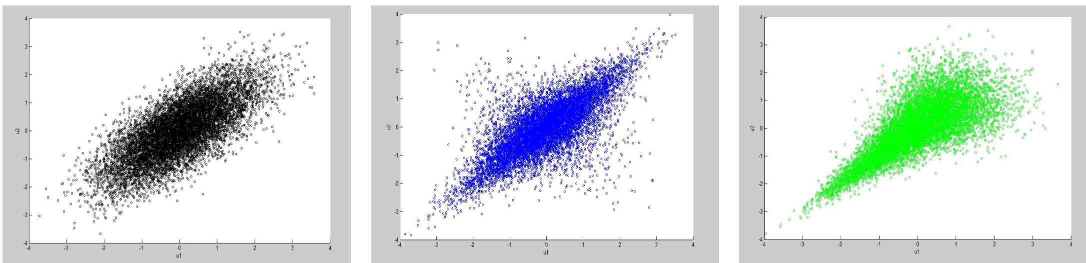
Proposition 4 (Calibration of copula parameters) *Based on a general proposition of Kendall's τ as a function of the copula C (Joe, 1997), the following relations between copula parameters and Kendall's τ hold:*

Copula	Kendall's τ
Gaussian	$\tau = \frac{2}{\pi} \arcsin(\rho)$
t	$\tau = \frac{2}{\pi} \arcsin(\rho)$
Clayton	$\tau = \frac{\alpha}{\alpha + 2}$

Figure 2 shows how the choice of the copula function influences realizations of a bivariate random vector with normally distributed marginals and a fixed correlation parameter of $\rho = 0.7$.

Figure 2: Realizations of a bivariate random vector under different copulas

The scatter plots are based on 10,000 realizations (simulated data pairs) under the Gaussian, the t_2 and Clayton copula, respectively, with standard normal marginals and consistent $\rho = 0.7$ in each case.



(a) Gaussian copula   (b) t_m copula with $m = 2$ (c) Clayton copula

2.2 The credit risk portfolio model

In this section of the paper, we describe the setup of the macroeconomic portfolio stress test measuring the impact of our stress scenario on regulatory capital ratios of German banks. Credit risk is described by a one-factor portfolio model based on Merton (1974) and VVasicek (2002) where the default of company i depends on the (latent) asset value Y_i. Y_i is a function of a sector-specific systematic risk factor and an idiosyncratic component, i.e.

$$Y_i = r \cdot X_{s(i)} + \sqrt{1 - r^2} \cdot U_i, \qquad r \in [0, 1] \tag{1}$$

with $s : \{1, ..., n\} \rightarrow \{1, ..., 17\}$ assigning one sector to each company. $X_{s(i)}$ is the systematic risk factor affecting company i pertaining to sector $s(i)$. The coefficient r is calibrated using an average of historic intersector correlations $\bar{\omega}$ and the standard average asset correlation for small and medium sized corporates $\bar{\rho} = 0.09$ as in Duellmann and Kick (2014). Following their approach we derive $\bar{\omega} = 0.79$ as the average of the correlation matrix $\hat{\Sigma}$ given in Appendix A.1 and set

$$r = \sqrt{\frac{\bar{\rho}}{\bar{\omega}}} = 0.34.$$

From now on, for simplicity of notation, the dependence of the sector on the company identifier will not be explicitly displayed, i.e. the systematic risk factor for sector s is denoted by X_s. The correlations between the risk factors are approximated by the sample correlations of sector stock index returns as suggested by Duellmann, Scheicher, and Schmieder (2008). We use weekly Eurostoxx stock index returns of the 17 sectors for the representative sample period from 1 January 2010 until 30 December 2011. As this period contains to a large extent the financial crisis, the estimated values are considerably impacted by the financial crisis and, therefore, they are higher than for other comparable studies. The estimated correlation matrix $\hat{\Sigma}$ can be found in Table A.1 in the appendix. The risk factors are then obtained by simulating the multivariate risk vector $X = (X_1, ..., X_{17})^T$, employing the respective copula for the interdependencies. In fixing each marginal distribution to be standard normal, the assumptions of the one-factor model with Merton background still hold and we can use different copulas to specify only the dependence structure between business lines.

The remaining parameters of the copula functions are obtained from the estimated correlation matrix $\hat{\Sigma}$. For determining m in the t_m copula, we fix $\hat{\Sigma}$ and use a maximum likelihood method based again on historical data from Euro Stoxx subindices and the copula density function $\lambda^t_{m,\hat{\Sigma}}$. To derive the parameter α for the Clayton copula, we make use of Proposition 4 and first calculate an average Kendall's τ out of $\hat{\Sigma}$ and then calibrate α conditioned on τ. It is very important to note that following this concept, Kendall's τ as a global measure of dependence is kept consistent and results are compared subject to copula functions.

In a nutshell, the stress test model works as follows: it is assumed that you have information about debtors of a corporate loan portfolio, i.e. you have estimates for the probabilities of default (PD), exposures at default (EAD) and a sector affiliation for each company. For the purpose of this study, LGDs are set to be constant at 0.45 (see Busch et al. (2015) for a detailed explanation of this ad hoc choice for the LGD parameter). First, calculate baseline risk ratios, such as expected loss (EL) and risk weighted assets (RWA), for the portfolio in a normal unstressed environment.

$$EL = \sum_{i=1}^{n} \frac{EAD_i}{EAD} \cdot LGD_i \cdot PD_i$$

The *Internal Ratings-Based Approach (IRBA)* allows for the following asymptotic description of RWAs for credit risk taking in a portfolio with n borrowers:

$$RWA_{CrR} = \sum_{i=1}^{n} \frac{EAD_i}{EAD} \cdot LGD_i \left[\Phi \left(\frac{\Phi^{-1}(PD_i^{stress}) + \sqrt{p\,(PD_i^{stress})}\,\Phi^{-1}(0.9999)}{\sqrt{1 - p\,(PD_i^{stress})}} \right) - PD_i^{stress} \right]$$

$$\cdot \left[\frac{1 + b\,(PD_i^{stress}) \cdot (T - 2.5)}{1 - 1.5 \cdot b\,(PD_i^{stress})} \right] \cdot 12.5 \cdot 1.06$$

with

$$p\,(PD_i) = 0.24 \cdot \left[1 - \frac{1 - \exp(-50\,PD_i)}{1 - \exp(-50)} \right] - 0.12 \cdot \left[\frac{1 - \exp(-50\,PD_i)}{1 - \exp(-50)} \right],$$

$$b\,(PD_i) = (0.11852 - 0.05478 \ln(PD_i))^2$$

and maturity $T = 2.5$.

In a second step, simulate risk vectors using different copula functions with calibrated parameters as explained. Take only those realisations which meet the conditions of the stress scenario, i.e. each risk component has to be less or equal to a specified stress threshold (Bonti et al., 2006). Then, plug the outcomes of the simulation into the one-factor model and get a number of firm values and a corresponding default barrier for each borrower in applying a reverse Merton approach, i.e. $\Phi^{-1}(PD_i)$. Calculate a stressed PD by taking relative frequencies and generate the stressed expected loss EL^{stress} and stressed risk weighted assets RWA^{stress} by just replacing PD with PD^{stress} in the formulas above. The impact of the stress scenario is then captured as the relation of unstressed and stressed characteristics. Moreover, stressed regulatory capital ratios, e.g. the $Tier1Capitalratio$, can be determined by means of a stress surcharge:

$$T1CR^{stress} = \frac{T1C - \frac{1}{2}\max\{EL^{stress} - TEP, 0\}}{RWA_{CrR}^{stress} + 12.5 \cdot (K_{MkR} + K_{OpR})}$$

with $T1C$ being the *Tier 1 Capital*, EL^{stress} being the expected loss under stressed conditions and TEP as the total of eligible provisions in accordance with Basel II. K_{MkR} and K_{OpR} represent regulatory capital requirements for unexpected losses from market and operational risks.

2.3 Macroeconomic scenario

With the setup of the portfolio model being illustrated, this section describes how a given stress scenario can be incorporated applying the modeling approach of Bonti et al. (2006). The stress impact is captured by restricting the distribution function of our sector-dependent systematic risk vector $X = (X_1, ..., X_{17})^T$ with a certain stress threshold for each component. In order to obtain these cutoff values and to link the latent unobservable variables of the sector-dependent systematic factors X_s to the historical stress scenarios from the observable GDP sector growth rates, we follow the steps described in Duellmann and Kick (2014) and Busch et al. (2015).

One of the most sensitive issues in macroeconomic stress testing is the question of scenario selection (e.g. Jandacka, Rheinberger, Breuer, and Summer, 2009). Since our main interest lies in the comparison of different copulas, we are content with a general stress scenario that can be considered both severe and plausible. In line with Busch et al. (2015), we apply a stress scenario that captures the experiences of the financial crisis in 2008/2009. Our scenario is however slightly more extreme in order to allow for a better analysis of the tail forecast of different copulas in stress testing. More precisely, we define the stress period as the core of the financial crisis from the third quarter of 2008 to the second quarter of 2009. The correlation structure of the risk factors ensures that all sectors are stressed in the scenario.

In order to specify the stress scenario, we calculate the geometric mean of the sector-specific GDP growth rates in the defined period. Using the historical development of the German GDP by sectors, we derive the sector-specific stress scenario. As the data on sectoral GDP breakdown are only available as of 1991 due to German reunification, it is difficult to estimate kernel densities on the basis of 21 years with 84 observations. In order to improve the estimation accuracy of the kernel densities, we obtain an enlarged sample of yearly sectoral GDP growth rates by bootstrapping techniques. The algorithm resamples the historical sectoral GDP growth rates and constructs yearly sectoral GDP growth rates by drawing from the quarterly historical sectoral GDP observations. In doing so, we obtain a robust sectoral GDP distribution. Compared with a flat GDP scenario assumption for all business sectors, our granular approach has the advantage that it enables us to exhibit more finely grained stress of the banks' sectoral credit portfolios, which were affected differently by the macroeconomic environment during the financial crisis.

Table 1: Cutoff values for the systematic risk factors

This table shows the cutoff values c_s describing the upper threshold of the stress region of the systemic risk factors in each sector.

ICB Classification	Cutoff Value c_s
Oil & Gas	0.27
Chemicals	-1.97
Basic Resources	-1.21
Construction & Materials	0.23
Industrial Goods & Services	-2.25
Automobiles & Parts	-2.08
Food & Beverages	-1.59
Personal & Household Goods	-2.08
Health Care	-2.16
Retail	-0.95
Media	4.26
Travel & Leisure	-2.24
Telecommunications	4.26
Utilities	4.26
Insurance	4.26
Financial Services	4.26
Technology	-2.32

As the business sectors are affected in different ways, the cutoff values c_s show a heterogenous stress impact across business sectors as Table 1 illustrates. The cutoff values are determined such that truncating the estimated kernel density of the sector-specific GDP growth rates at the cutoff value results in a conditional expectation that corresponds to the observed sectoral growth rate from the third quarter of 2008 to the second quarter of 2009. Business sectors such as industrial goods and services as well as technology are heavily stressed whereas financial services or utilities sectors are not influenced by the stress scenario.[5]

Following this approach, 12 out of 17 sectors are directly stressed in truncating the multivariate distribution function (a threshold of 4.26 is not a real truncation for a standard normal variable). The impact on all other branches is captured via dependencies of risk components using the copula concept.

[5]The result of no stress in the financial sector in the crisis is surprising; however, this is warranted by the data on which the estimations are based. The financial services subsector of the German GDP decreased only slightly during the stress period and remained on a relatively high level compared to e.g. the period from 2002 until 2005, during which it saw a huge decline.

2.4 Data and descriptive analysis

The models used require input data on the portfolio composition of the analyzed banks, borrower credit quality and on the sector correlation structure. The reference date of our stress test is 31 December 2011. The information about the portfolio composition and the borrowers is based on the German credit register which is hosted by the Deutsche Bundesbank and includes all national and international borrowers with a minimum total credit volume of €1.5 mn. The term “borrower” in this context includes not only single borrowers but also so-called borrower units which can comprise several formally independent but (legally or economically) heavily interlinked entities.[6] For each single entity, borrower information on the loan volume, the PD and the sectoral “Nomenclature statistique des activités économiques dans la Communauté européenne” (NACE) code is available in this data base.

As for the sectors, the NACE codes of the respective borrowers are aggregated to supersectors as defined by the Industry Classification Benchmark (ICB) that concur with Standard & Poor’s Eurostoxx sectoral subindices used for estimating the intersectoral correlations.

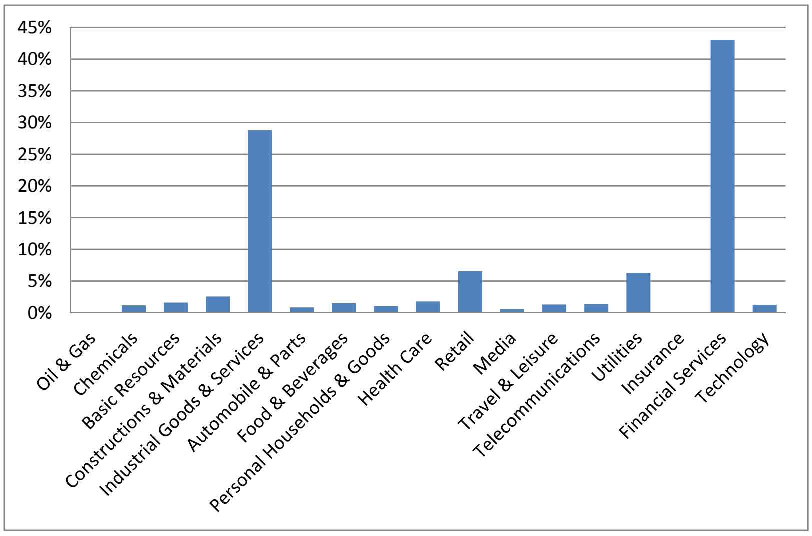
Figure 3 displays the sectoral distribution of the loans. The major borrowing sectors are industrial goods and services and financial services.

For the stress forecast output, three key measures are calculated: RWA, EL and regulatory capital ratios. Since the data contain only a sample of each bank’s portfolio, we calculate the overall effect on these risk measures by combining the stress forecasts with the respective figures in the German solvency reporting. The effect on RWA is calculated by multiplying the total RWA for the corporate portfolio, i.e. RWA as treated under the standardized approach and under the bank’s IRB approach, with the relative change of RWA from the respective bank’s stress scenario. Furthermore, the RWAs under the standardized approach increase due to the higher risk weights for defaulted exposures, which affects the denominator of the capital ratio.

[6] Borrower units are defined as a group of single borrowers which can comprise several formally independent but (legally or economically) heavily interlinked entities. For the borrower units, however, PD and NACE are not contained in the data of the German credit register and need to be identified first. For the mapping of the NACE code we use the NACE code of the sector with the highest loan amount from within the total loan volume of all banks to this borrower unit as this information does not depend on the situation in the respective bank. The PD of a borrower unit is calculated by a weighted average of all loans of the respective bank to the single borrowers within this borrower unit. Where no PD on single borrower basis is available we use the same concept as described for the single entities taking into account the bank’s sectoral average PD or total bank average PD for the respective single borrower.

Figure 3: Distribution of loan volume per sector

This bar chart shows the percentage of loan volume in the sample for each ICB sector. The figures are based on the NACE code of the borrowers in the German credit register and aggregated to supersectors that concur with the ICB classification.



Due to the higher PDs under stress, the EL increases in the stress scenario. This only affects the exposures treated under the IRB approaches because the calculation according to the standardized approach does not include expected losses as these are already covered by the consideration of specific provisions when calculating the regulatory capital. Furthermore, the EL for defaulted exposures is also not taken into account because the PD is already equal to one and therefore the EL cannot increase anymore. The effect on the EL for non-defaulted IRB exposures is calculated by multiplying the change in EL per bank with the bank's EL prior to the stress scenario.

The regulatory capital of the banks is affected by the capital requirements framework. All banks have to calculate the excess or shortfall of provisioning over the EL. A shortfall of provisions will be deducted from capital. To calculate the effect on the regulatory capital, we deduct the increase in the EL from regulatory capital. In the case that a part of the excess of provisions over the EL is not used as Tier 2 capital prior to the stress calculation, we deduct only the part that is not covered by these unrecognized excesses. This deduction will be taken 50 percent from Tier 1 capital and 50 percent from Tier 2 capital. If there is not enough Tier 2 capital to cover the respective EL deduction, the exceeding amount will be recognized as an additional deduction from Tier 1 capital. The calculated amount of Tier 1 and Tier 2 capital after stress will be used to calculate the effect on the capital ratios. The effect of the stress scenario on the capital ratios of the banks is calculated by using the capital after stress and the RWA after stress as described above. We calculate the Tier 1 capital ratio and the total capital ratio by dividing the respective amount of capital after stress by the stressed RWA. A bank is considered to fail the stress test if the Tier 1 capital ratio is below 4 percent. The descriptive statistics of the applied bank sample are shown in Table 2.

Table 2: Descriptive statistics of bank sample

This table shows key figures of our stress test data set. Unless specified differently, all numbers are composites from the amounts measured under the standardized approach (SA) and the IRB approach.

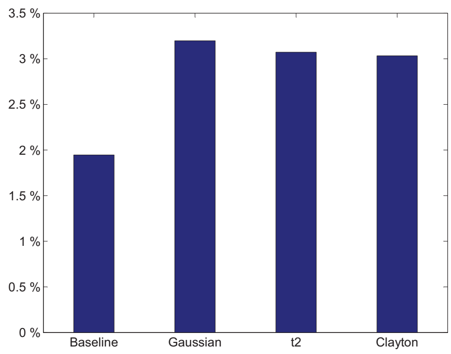
Variable	Amount (in € bn. or %)
Number of banks	17
Total assets	5871.8
Percentage of total assets of all German banks	58.9%
Total credit exposure (corporates)	1,489.00
EL per credit exposure (corporates), IRB only	1.9%
Total RWA	1,419.90
Total RWA (corporates)	689.72
Total RWA per total credit exposure (corporates)	46.3%
Total Tier 1 capital	175.9
Tier 1 capital ratio	12.4%

3 Bank stress test results

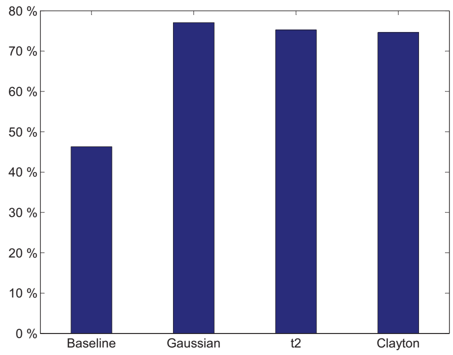
In this section we describe the stress impact on EL, RWA and regulatory capital ratios using our stress testing framework subject to different copula functions. Figure 4 further illustrates the levels of EL, RWA and regulatory capital ratios in the baseline and the stress scenario using the different copula functions. EL in relation to credit exposure values is forecast to raise from 1.9% to 3.2%, 3.1% and 3.0% using the Gaussian, the t_2 and the Clayton copula, respectively. RWAs per exposure increase from 46.3% to 77.1%, 75.3% and 74.6% for the respective copulas, reflecting the procyclical characteristics of the measurement of RWA. Even though these numbers by themselves merit attention, our main interest here lies in the comparison of the stress forecasts using different copulas.

Figure 4: Stress impact on Expected Loss, Risk Weighted Assets and Regulatory capital ratios

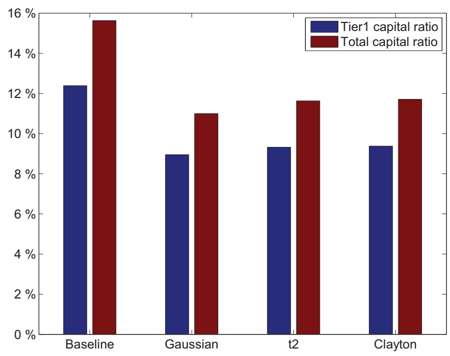
These bar charts show the forecasts of key portfolio variables such as EL, RWAs and regulatory capital ratios as a percentage of exposure. Baseline refers to the unstressed values, Gaussian, t_2 and Clayton to the stress scenario forecast using the Gaussian copula, the t_2 copula and the Clayton copula, respectively.



(a) Expected Loss



(b) Risk Weighted Assets



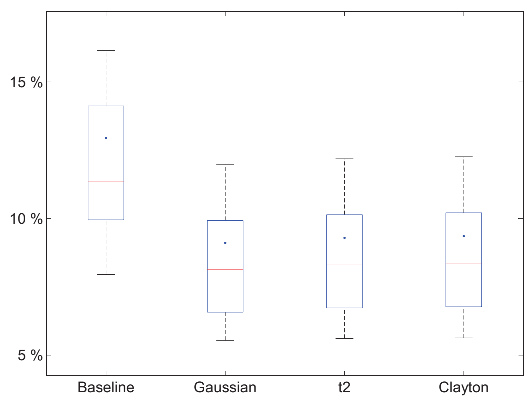
(c) Regulatory capital ratios

As can be seen from the output, German banks are forecast to weather the stress scenario relatively well, with the weighted average Tier 1 capital ratios consistently above nine percent in the stress scenarios. What makes the results striking is, however, that the state of banks' capital ratios is forecast to be worse under the Gaussian copula than using the t_2 copula or the Clayton copula. The employed methodology does not yet allow for an indication of uncertainty around the forecasts, so it is not possible to comment on the statistical significance of the difference between the predictions. However, a comparison of the three stress forecasts provides a crude indication of the extent of the difference between the Gaussian copula and the other approaches: e.g. for EL, the difference between the Gaussian and the t_2 copula forecast, which is the forecast closest to the Gaussian one, is more than three times larger than the difference between the t_2 and the Clayton forecast. Even though the differences in terms of capital ratio changes appear not to be considerable, transforming them into concrete capital positions, these differences can be material. This implies that the greater severity of the Gaussian forecast cannot be easily dismissed as a chance phenomenon but rather merits a more in-depth analysis.

Figure 5 displays more fine-grained information on the stress forecasts of the Tier 1 capital ratios. Again, the results clearly show that the Gaussian copula gives a more severe stress forecast than heavy-tailed copulas.

Figure 5: Distribution of Tier 1 capital ratios under normal and stressed conditions

This box plot depicts the variation of the Tier 1 capital ratio forecasts across the individual banks in the sample. Baseline refers to the unstressed values, Gaussian, t_2 and Clayton to the stress scenario forecast using the Gaussian copula, the t_2 copula and the Clayton copula, respectively. The upper and lower limits of the boxes are the 75% quantile (q_3) and the 25% quantile (q_1). The middle horizontal line is the median and the single point in the box represents the mean. The two whiskers are the most extreme data points not considered to be outliers. Points are drawn as outliers if they are larger than $q_3 + 1.5 \cdot (q_3 - q_1)$ or smaller than $q_1 - 1.5 \cdot (q_3 - q_1)$. The box plots for capital ratios are shown without outliers.



We hypothesize that this obtained result stems from our stress test setup with a high correlation structure between business sectors and a severe stress scenario where more than two thirds of the components lie under a given stress threshold. Investigating the variables of the model, we see that the difference between the stress forecasts originates from the simulated risk factors in the credit risk model: the mean of the risk vector is -2.83 in the Gaussian, -2.74 in the t_2 copula and -2.73 in the Clayton case. Since these risk factors have a strong impact on probabilities of default of the entities in the credit portfolio, the risk indicators of our stress test are affected in equal measure. If we choose another less severe stress scenario with only three stressed sectors, as in the setup in Duellmann and Kick (2014), but keep the correlation matrix fixed, the averages of simulated risk vectors are, in the same order as previously, -2.39, -2.57 and -2.80. Hence, with this kind of scenario setup, the Clayton copula delivers the adequate stress forecast whereas in the Gaussian case, the stress impact is much lower. It therefore seems that the phenomenon of greater severity using the Gaussian copula is related to the number of truncated risk factors and their particular cutoff level. In the next section, this question will be further investigated by examining the expected values of the risk factors under variations of the input parameters for the stress scenarios applying simulation algorithms: the correlation matrix, the number of stressed factors and the severity of the stress scenario characterized by cutoffs.

4 Simulation study of input parameters

In order to explain the results of Section 3, we derive precise results on the expected values of the risk vector in our stress testing methodology here. The reason why we focus on the risk vector is that it is the main driver of stress in the setup of the model since the idiosyncratic components U_i in equation (1) are modeled as an *i.i.d.* white noise process that, on aggregate, cannot account for systematic differences. The results of this section apply more generally to random variables linked by copulas. However, for illustrative purposes, we will still refer to X as the systematic risk vector and the cutoff value c as the stress threshold. For the analysis to be feasible, we restrict the risk vector to be two-dimensional for the first part of this section.

With $X = (X_1, X_2)$, $X_1 \sim F_1$, $X_2 \sim F_2$ and $X \sim F = C(F_1, F_2)$, the target measure is the conditional expectation of the random variable $\bar{X}$ representing the average of two components X_1 and X_2, i.e.

$$\bar{X} = \frac{1}{2}X_1 + \frac{1}{2}X_2$$

and

$$\mathbb{E}^C[\bar{X}|X_1 \leq c_1, X_2 \leq c_2] = \frac{1}{2}\mathbb{E}^C[X_1|X_1 \leq c_1, X_2 \leq c_2] + \frac{1}{2}\mathbb{E}^C[X_2|X_1 \leq c_1, X_2 \leq c_2]$$

due to the linearity of conditional expectations. With the notation $\mathbb{E}^C$ we stress that the conditional expectation, and above all, its outcome are determined by the choice of copula function to model dependence between the risk components. In the following, we discuss the impact of copulas for both homogenous and heterogenous stress effects.

4.1 Homogeneous stress effect

For the homogeneous stress case where $c_1 = c_2 = c$ and X_1, X_2 are uniformly distributed, it holds that

$$\mathbb{E}^C[X_1|X_1 \leq c_1, X_2 \leq c_2] = \mathbb{E}^C[X_2|X_1 \leq c_1, X_2 \leq c_2] = \mathbb{E}^C[\bar{X}|X_1 \leq c_1, X_2 \leq c_2]$$

and it is sufficient to compute the conditional expected value of X_2.

Let u_1, u_2 be two uniformly distributed random variables and λ^C the density of the copula function which models the joint distribution of u_1 and u_2. Then, the two-dimensional random vector $X = (X_1, X_2)$ with standard normal marginals can be written as $(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$ (see Chapter 3) and we can calculate

$$\begin{aligned}
&\mathbb{E}^C[X_2|X_1 \leq c_1, X_2 \leq c_2] = \int_{-\infty}^{c_2} x_2 \cdot \mathbb{P}(X_2 \in dx_2 | X_1 \leq c_1, X_2 \leq c_2)\ dx_2 \\
&= \int_{-\infty}^{c_2} x_2 \cdot \frac{\mathbb{P}(X_2 \in dx_2, X_1 \leq c_1)}{\mathbb{P}(X_1 \leq c_1, X_2 \leq c_2)}\ dx_2 \\
&= \frac{1}{\mathbb{P}(u_1 \leq \Phi(c_1), u_2 \leq \Phi(c_2))} \int_{-\infty}^{c_2} x_2 \cdot \mathbb{P}(u_2 \in d\Phi(x_2), u_1 \leq \Phi(c_1))\ \varphi(x_2)\ dx_2 \\
&= \frac{1}{\mathbb{P}(u_1 \leq \Phi(c_1), u_2 \leq \Phi(c_2))} \int_{-\infty}^{c_2} x_2 \cdot \int_{0}^{\Phi(c_1)} \lambda^C(x_1, \Phi(x_2))\ dx_1\ \varphi(x_2)\ dx_2 \qquad (2)
\end{aligned}$$

The inner integral must be solved numerically because integrating $\Phi(\cdot)$ is not analytically tractable. For the Clayton copula, the conditional expectation can be rewritten explicitly

$$\begin{aligned}
&\mathbb{E}^{Clayton}[X_2|X_1 \leq c_1, X_2 \leq c_2] = \frac{1}{(\Phi(c_1)^{-\alpha} + \Phi(c_2)^{-\alpha} - 1)^{-1/\alpha}} \\
&\cdot \int_{-\infty}^{c_2} x_2 \cdot \lim_{\epsilon \to 0} \Big(\big(\Phi(c_1)^{-\alpha} + \Phi(x_2)^{-\alpha} - 2\big)^{-\frac{\alpha+1}{\alpha}} - \big(\epsilon^{-\alpha} + \Phi(x_2)^{-\alpha} - 1\big)^{-\frac{\alpha+1}{\alpha}} \Big)\ \varphi(x_2)\ dx_2
\end{aligned}$$

which then has to be solved numerically.

For the Gaussian copula, the expression for the expected value of a two-dimensional normally distributed random variable can be employed

$$\mathbb{E}^{Ga}[X_2|X_1 \leq c_1, X_2 \leq c_2] = \frac{1}{\Phi_2(c_1, c_2)} \int_{-\infty}^{c_2} \int_{-\infty}^{c_1} x_2 \cdot \varphi(x_1, x_2)\ dx_1\ dx_2,$$

which can also be solved numerically.

For the t_2 copula, the density of a t_2 copula as described in Definition 3 can be plugged into Equation 2, such that

$$\begin{aligned}
&\mathbb{E}^{t_2}[X_2|X_1 \leq c_1, X_2 \leq c_2] = \frac{1}{\mathbb{P}(u_1 \leq \Phi(c_1), u_2 \leq \Phi(c_2))} \\
&\cdot \int_{-\infty}^{c_2} x_2 \cdot \int_0^{\Phi(c_1)} \frac{\left(1 + \frac{t_2^{-1}(x_1)^2 + t_2^{-1}(\Phi(x_2))^2 - 2\rho t_2^{-1}(x_1) t_2^{-1}(\Phi(x_2))}{2\sqrt{(1-\rho^2)}}\right)^{-2}}{\sqrt{(1-\rho^2)}\Gamma\left(\frac{3}{2}\right)^2 \left(1 + \frac{t_2^{-1}(x_1)^2}{2}\right)^{-\frac{3}{2}} \left(1 + \frac{t_2^{-1}(\Phi(x_2))^2}{2}\right)^{-\frac{3}{2}}}\ dx_1\ \varphi(x_2)\ dx_2
\end{aligned}$$

where the double integral needs to be solved numerically.

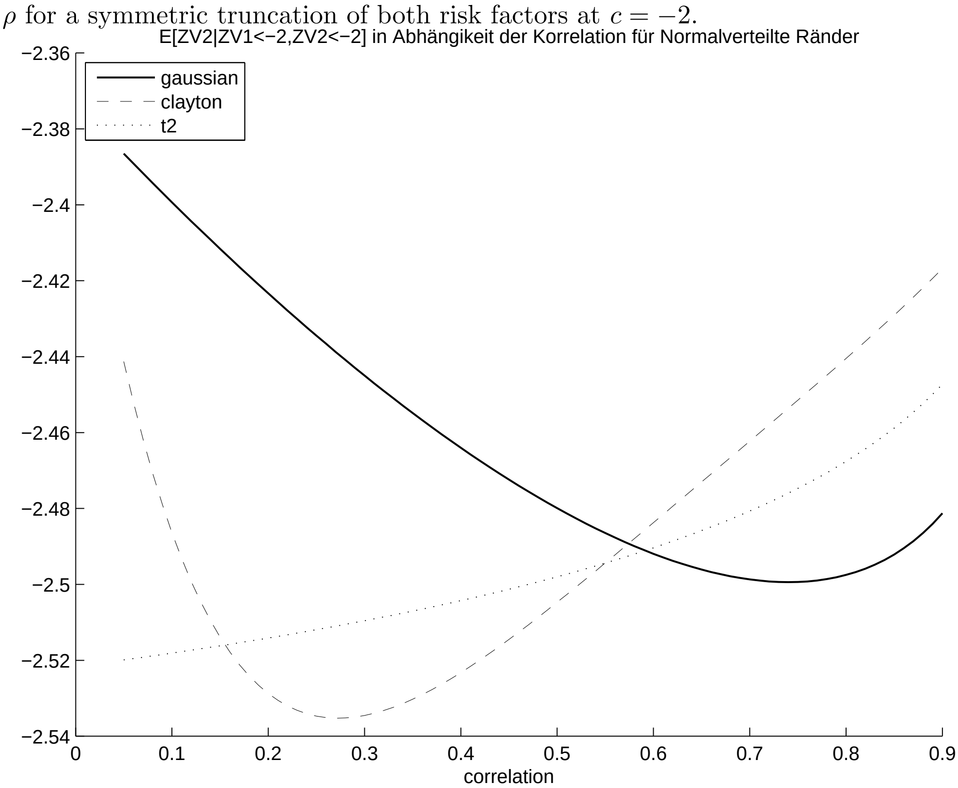
Applying the formulas derived here, we now analyse how the simulation input parameters in our stress test setup, i.e. correlation and cutoff values, influence our quantity of interest for each choice of copula function.

4.1.1 Impact of the degree of correlation

Figure 6 shows the conditional expected value of $\bar{X}$ as a function of the correlation value of the two risk components for a given cutoff value $c = -2$ which represents a severe stress level. In general, the economic impact of the difference is not that large as it amounts up to ten percent. When correlation is weak, the t_2 copula generates the most severe results, for moderate correlation values of approximately 15% to 50%, the Clayton copula yields the lowest expected value of the risk factor meaning the strongest stress effect. As one can observe here, there is something like a turning point where for higher degrees of correlation, i.e. correlations greater than 60%, the Gaussian copula implies the highest level.

Figure 6: Impact of the degree of correlation on the conditional expected value of $\bar{X}$

This figure displays the impact of a change in the degree of correlation on the relative severity of the stress forecast under different copulas, leaving everything else equal. The horizontal axis displays the degree of correlation ρ, whereas the vertical axis shows the expected value of the risk factor as a function of ρ for a symmetric truncation of both risk factors at $c = -2$.



The correlation value is one of the main drivers of our results discussed in Section 3. This property of the conditional expectations of the systematic risk factors is very striking and counterintuitive at first glance, but it can be explained by the truncation of risk components which reverses general intuition associated with copulas. Next, we investigate the dependence of the expected value of the risk factor $\bar{X}$ on both the degree of correlation and on the cutoff level.

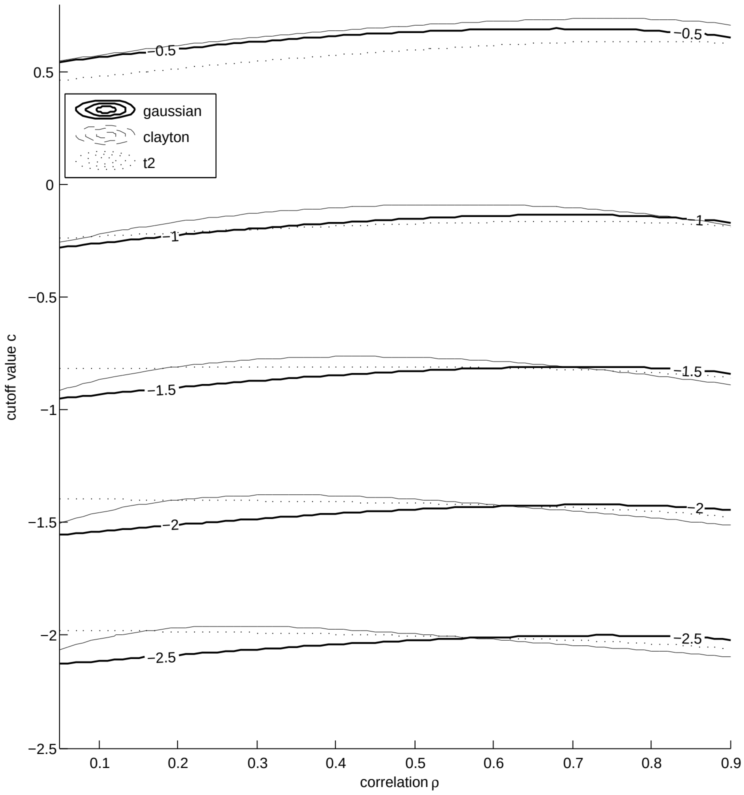
4.1.2 Impact of cutoff level and of the degree of correlation

Figure 7 plots the level curves of the expected values of $\bar{X}$ as functions of the cutoff level c and the degree of correlation, i.e. all points (c, ρ) for which $\mathbb{E}^C[\bar{X}|X_1 \leq c, X_2 \leq c]$ is equal to i, with $i \in \{-0.5, -1.0, -1.5, -2.0, -2.5\}$. The plotted curves therefore show which configuration of the input parameters is necessary to generate a given level of stress. If, for the same level of stress, the level curve for one copula lies above the one for another copula, the former copula can be considered as more severe since then, on average, the same stress level will be generated under a less strict truncation of the distribution of the risk factor.

Examining Figure 7, it becomes clear that the relative severity of the different copulas depends on the configuration of the input parameters and that one general rule does not apply. For low degrees of correlation and small i, the level curves for the Gaussian copula lie below those of the other two, which is in line with the intuition of the Gaussian copula being the least severe. However, the result reverses when other input parameter configurations are made. First, for $\mathbb{E}^C[\bar{X}|X_1 \leq c, X_2 \leq c](c, \rho) = -0.5$, i.e. for a low level of stress, the Gaussian copula is more severe than the t_2 copula. Second, and more importantly, in more extreme stress scenarios, the Gaussian copula becomes more severe than the others when the correlation value between the two risk components increases, which can already be seen from Figure 6. Figure 7 shows that there is also an interaction effect between the cutoff value and the degree of correlation at which the Gaussian copula turns more severe: the smaller the cutoff value, the smaller the degree of correlation at which the Gaussian and the Clayton level curves intersect, which is the degree of correlation at which the Gaussian copula becomes more severe than the Clayton copula.

Figure 7: Impact of cutoff level and the degree of correlation on the conditional expected value of $\bar{X}$

This figure displays the level curves of the conditional expectation of $\bar{X}$ for different values of c as a function of the degree of correlation. The impact on the relative severity of the stress forecast is analyzed for different copulas, leaving everything else constant. More precisely, the figure shows all points (ρ, c) for which $\mathbb{E}^C[\bar{X}|X_1 \leq c, X_2 \leq c] = i$, $i \in \{-0.5, -1.0, -1.5, -2.0, -2.5\}$.



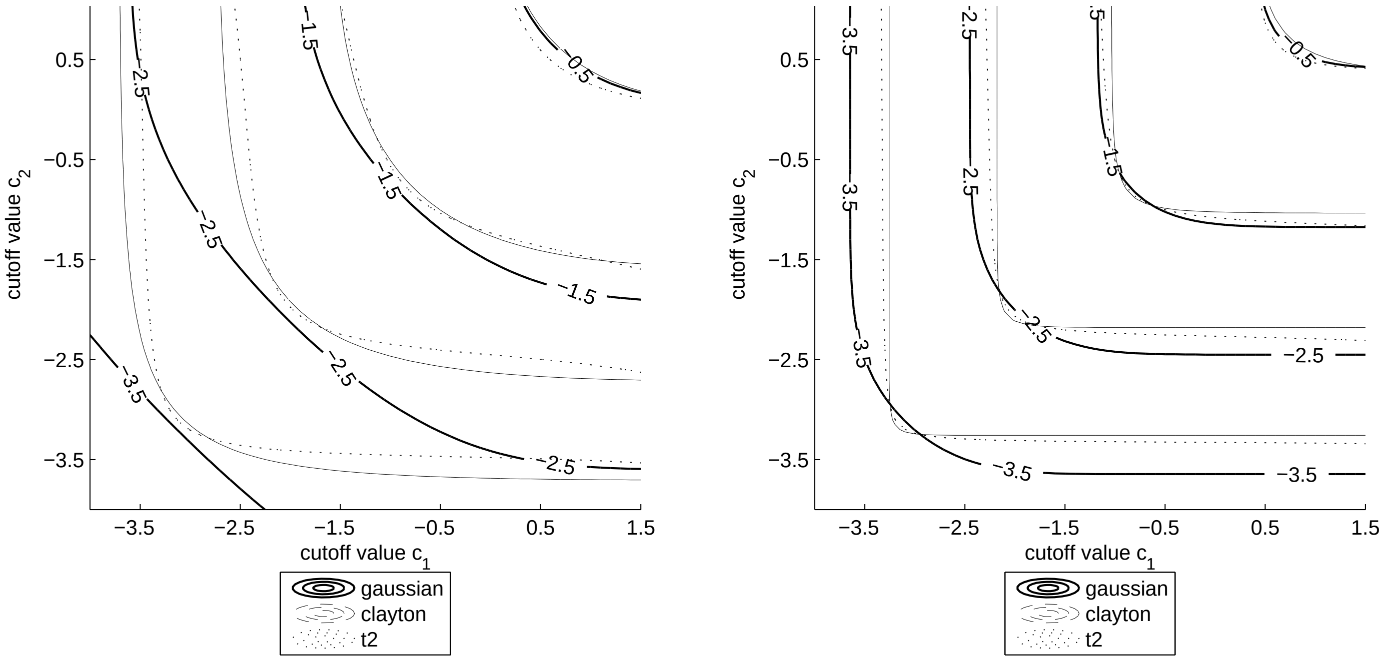
These results indicate that there is no general rule for the behavior of risk factor $\mathbb{E}^C[\bar{X}|X_1 \leq c_1, X_2 \leq c_2]$ dependent on the copula function, the underlying correlation and the cutoff levels (even in the case of homogeneous stress). Assumptions regarding certain properties of copulas might not hold true given specific stress constellations, such that in order to determine the copula function creating the most severe results, one might have to test different copula models. Next, we relax the simplifying assumption of $c_1 = c_2$.

4.2 Heterogeneous stress effect

In situations where the cutoffs for X_1 and X_2 vary, looking only at the conditional expected value of X_2 is not sufficient such that we have to calculate conditional expected values for both of the components of our risk vector. The corresponding formula for X_1 is derived in analogy to the formulas given in the first paragraphs of Section 4.2. Figure 8 again plots the level curves of the conditional expectation of the random variable $\bar{X}$, but now for a fixed degree of correlation and as a function of the two cutoff values, c_1 and c_2. It therefore illustrates how the results change when the risk factors X_1 and X_2 are truncated at different levels, i.e. when the stress on the risk factors is heterogeneous.

Figure 8: Impact on the conditional expected value of $\bar{X}$ for different values of c_1 and c_2

This figure displays the impact of a change in either of the two cutoff values on the relative severity of the stress forecast under different copulas, leaving everything else constant. More precisely, the figure shows all points (c_1, c_2) for which $\mathbb{E}^C[\bar{X}|X_1 \leq c_1, X_2 \leq c_2] = i$, $i \in \{-0.5, -1.5, -2.5, -3.5\}$. The left subfigure displays the relationship for $\rho = 0.3$ whereas the right subfigure does the same relationship for $\rho = 0.8$.



From the slopes of the level curves it can be seen that, except for mild stress conditions, our quantity of interest $\mathbb{E}^C[\bar{X}|X_1 \leq c_1, X_2 \leq c_2]$ is more responsive to a change in one of the cutoff levels while the other cutoff level remains fixed under the Gaussian copula, which is probably due to the Gaussian copula exhibiting no tail dependence. As ρ increases, the level curves approach an "L"-shape, the ones for the t_2 and the Clayton copula at an even faster pace than the Gaussian one. For c_1 close to c_2, the Gaussian level curve then lies above the other two. Consequently, the phenomenon of the Gaussian copula giving more severe stress forecasts is specific to the case of high correlation of the risk factors and relatively homogenous stress.

4.2.1 Impact of number of cutoffs

Besides the analysis on correlation and cutoff values, another important issue concerns the relationship between the expected values of $\bar{X}$ using different copula approaches and the number of truncated risk factors compared to the total number of risk components. We need to analyze higher dimensional risk vectors which is important, in particular, for large stress test exercises considering detailed breakdowns of systematic factors such as country or business sector in order to investigate this relationship. Since clear formulas for the conditional expectations of $\bar{X}$ can only be derived for the two dimensional case, we perform Monte Carlo simulations extending the sample to five and ten business sectors.[7] The number of simulations for each configuration of the data generating process is $N_{sim} = 10,000$. A configuration refers to the choice of copula, the degree of correlation, $\rho \in \{0.3, 0.8\}$, the number of total risk factors, $N_X \in \{5, 10\}$, and the number of truncated or stressed factors, $N_X^{stressed} \in \{1, 2, \ldots, N_X\}$. Each stressed factor is truncated at a fixed level of $c = -2$ which implies a relatively severe level of stress. Figure 9 shows the average realizations of the first stressed risk factor for different copulas[8].

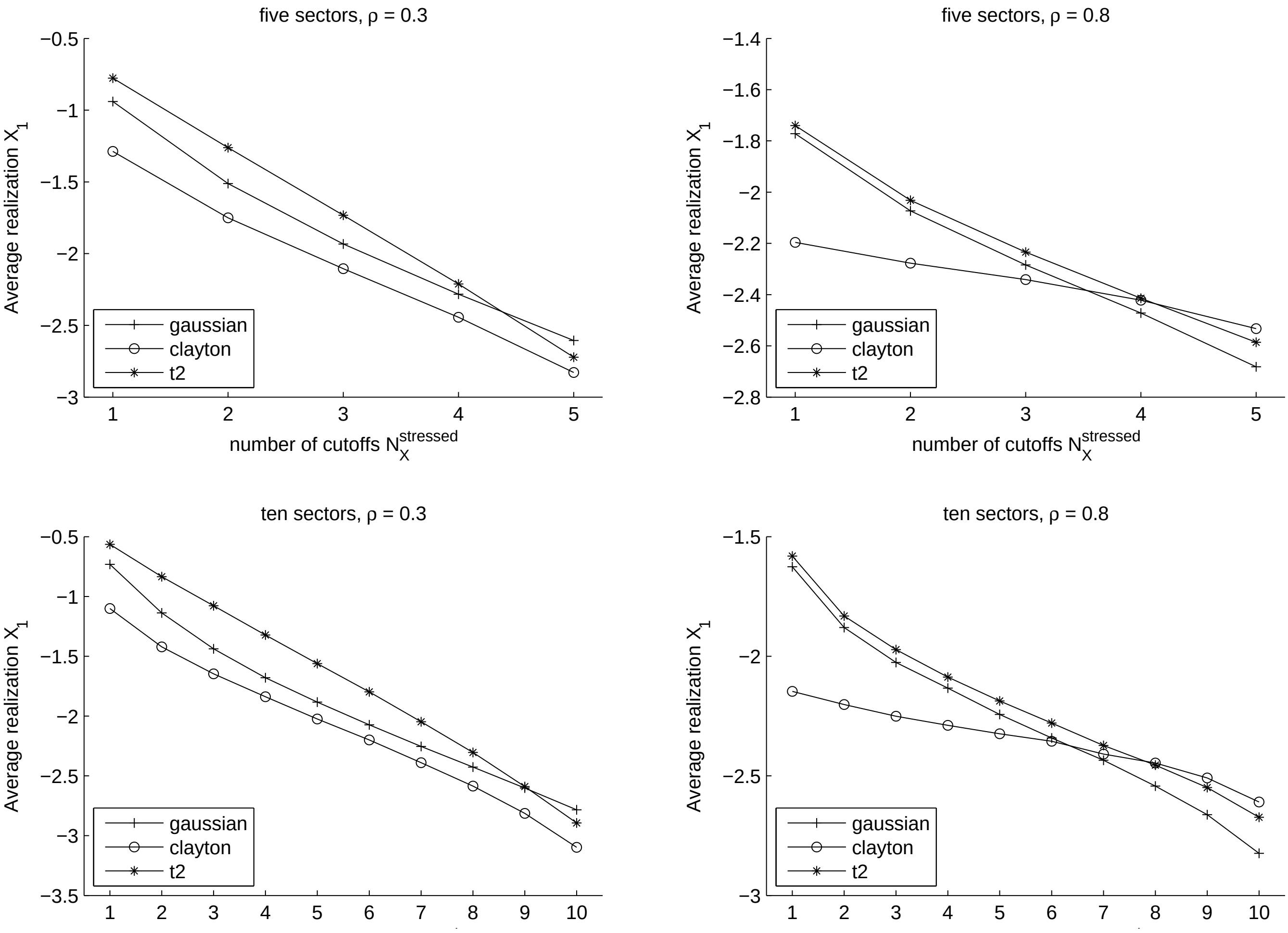
Examining Figure 9, we find that the Gaussian copula gives more severe stress forecasts than the other two copulas for high ρ and high ratio $N_X^{stressed}/N_X$. The striking feature of the figure is that it implies that the higher severity of the Gaussian copula occurs only if a large part of the total number of risk factors is stressed: As can be seen in Figure 6, for $N_X = N_X^{stressed} = 2, \rho = 0.8$, the Gaussian copula obtains the most severe stress forecasts, for the same degree of correlation and the same cutoff value, this is the case only if $N_X^{stressed} \geq 4$ for $N_X = 5$ and $N_X^{stressed} \geq 7$ for $N_X = 10$. The number of stressed risk factors is another condition under which the counterintuitive result of higher severity of the Gaussian copula holds: A large group (more than 70-80%) of the risk factors need to be severely stressed.

[7] The standard errors of the Monte Carlo simulations are very low. The largest single calculated standard error is 0.009 and, therefore, does not impact on the estimations.

[8] For this analysis, the setup equals again that of a homogeneous stress case with standard normal marginals; therefore it is sufficient to limit the estimation to a single random variable X_1 in order to measure the behavior of $\bar{X}$.

Figure 9: Impact of a different number of stressed factors

This figure displays the average realization of the first stressed risk factor as a function of the number of truncated risk factors, $N_X^{stressed} \in \{1, 2, \ldots, N_X\}$, using different copulas. The upper two and the lower two subfigures show this for a total number of risk factors $N_X = 5$ and $N_X = 10$, respectively. The left column assumes $\rho = 0.3$, the right one $\rho = 0.8$. The number of simulations is $N_{sim} = 10,000$ and the cutoff value is set to $c = -2$ in each case.



In this section, we have shown that the higher severity of the Gaussian copula under normal marginals in a stress scenario is not just a chance result but a feature of the conditional expectation of the systematic risk factors. We have also identified four conditions under which the result of a more severe Gaussian copula holds: high correlation between the risk factors, high and homogeneous stress and a large proportion of stressed risk factors.

5 Implications for stress testing

Copulas are an indispensable tool for modeling multivariate dependencies, in stress testing as well as in other areas of risk management. The use of the Gaussian copula has often been heavily criticized for downplaying interdependencies compared to other copulas with higher tail dependence. In this paper, we show that the latter is not necessarily true. However, to choose the appropriate copula for credit risk stress testing, multiple criteria have to be taken into account. No copula can be classified a priori as the best selection. It would be advisable to investigate a variety of dependence structures and determine which specification leads to the most severe stress forecast. The specification of the stress scenario is normally exogenously determined and based on macroeconomic information. More precisely, in our stress testing framework, the stress scenario specifies the correlation value between the business sectors, the direct stress level via cut off values, the stress distribution across risk factors (homogenous/heterogenous) and the number of stressed business sectors. Against this background, the only remaining degree of freedom is the choice of dependence modeling meaning the selection of the copula function. As our simulation study reveals, this choice can impact considerably on the banks' capital ratios or other obtained figures.

The Gaussian copula is able to generate severe stress scenarios when assuming extreme stress forecasts which outweigh the effects of the Clayton or t copula. More precisely, if the determined stress scenario is characterized by very low cut off values for many business sectors and high sector correlation values possibly combined with a homogenous stress distribution across the affected sectors, the Gaussian copula would be an appropriate choice for estimating high stress effects. The reason for this is that the Gaussian copula is an elliptical distribution for which (joint) extreme events are less likely when considering the entire distribution but more probable when limited to a very small part of its tail. In general, other light-tailed copulas such as the Frank copula could also be suitable. Nevertheless, as the Gaussian copula is still the industry-standard today and can be easily applied for stress testing, alternatives are only recommended when additional restrictions are present.

In case of less severe adverse scenarios, either the Clayton or the t copula would be the recommended copulas. The Clayton copula as an asymmetric distribution is characterized by strong tail dependence on one side which means that it is a heavy-tailed copula. Against this background, it is possible to estimate rather high stress levels in environments with lower correlation values and only a few stressed business sectors, i.e. spill-over effects are then captured very well.

The t copula, like the Gaussian copula, belongs to the elliptical distributions and is accordingly symmetric. The level of tail dependence is lower than for the Clayton copula, but it is modeled at both tails. For very low correlation values, meaning for weak stress scenarios, the t copula generates comparably high stress levels. With regard to its similarity to the Gaussian copula, the t copula always represents the first alternative for using the Gaussian copula as this copula can easily replace the other. However, in a number of cases, the considered heavy-tailed copulas generate stress levels which are close to each other. Special attention has to be paid to situations with a limited number of stressed sectors in which the Clayton copulas considerably outperform the elliptical copulas.

The conditions for the severity of the Gaussian copula to hold true are not as restrictive as they may sound: our study shows that these situations might easily arise in practice. Our results intend to raise awareness regarding possible counterintuitive effects when designing stress test frameworks and conducting top-down stress test exercises. In particular, our findings could be beneficial for banks running their own internal stress tests as well as for regulators and market analysts. Implementing the adequate severity for the applied stress scenarios leads to a better assessment of internal risk structures and identification of impacted business areas. Furthermore, some topics, such as concentration risk, which have only sparsely been considered in stress test exercises so far, can be incorporated in stress testing frameworks to more properly account for relevant side effects. Furthermore, quality assurance processes for bottom-up stress tests can benefit from using adequate copulas when implementing assumptions on underlying dependence structures.

Our paper shows that future work on the behavior of copula functions in unusual circumstances, in particular in stress testing, might be a fruitful endeavor. As one example, the modeling of higher dimensional risk vectors using vine copulas warrants further attention.

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A Appendix

A.1 Correlation matrix

Table A.1: Correlation matrix of the sector indices

This table shows inter-sectoral correlations of 17 sector indices following the ICB sector classification. The correlations were estimated from weekly stock index returns from 1 January 2010 until 30 December 2011.

	Sector	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	Oil and Gas	1	0.84	0.86	0.89	0.87	0.76	0.69	0.83	0.74	0.79	0.89	0.78	0.82	0.87	0.89	0.87	0.80
2	Chemicals	0.84	1	0.87	0.88	0.91	0.84	0.74	0.89	0.77	0.81	0.83	0.79	0.73	0.78	0.82	0.86	0.77
3	Basic Resources	0.86	0.87	1	0.90	0.93	0.83	0.62	0.83	0.66	0.77	0.85	0.80	0.69	0.78	0.82	0.87	0.8
4	Construction and Materials	0.89	0.88	0.90	1	0.93	0.81	0.69	0.86	0.71	0.84	0.90	0.82	0.80	0.85	0.90	0.90	0.82
5	Industrial Goods and Services	0.87	0.91	0.93	0.93	1	0.90	0.70	0.90	0.73	0.85	0.90	0.85	0.77	0.83	0.86	0.92	0.851
6	Automobiles and Parts	0.76	0.84	0.83	0.81	0.90	1	0.60	0.86	0.69	0.80	0.79	0.77	0.65	0.72	0.75	0.84	0.74
7	Food and Beverage	0.69	0.74	0.62	0.69	0.70	0.60	1	0.77	0.71	0.73	0.75	0.70	0.67	0.65	0.66	0.65	0.63
8	Personal and Household Goods	0.83	0.89	0.83	0.86	0.90	0.86	0.77	1	0.74	0.85	0.85	0.82	0.71	0.74	0.78	0.85	0.78
9	Health Care	0.74	0.77	0.66	0.71	0.73	0.69	0.71	0.74	1	0.75	0.75	0.71	0.66	0.66	0.71	0.72	0.69
10	Retail	0.79	0.81	0.77	0.84	0.85	0.80	0.73	0.85	0.75	1	0.85	0.80	0.76	0.78	0.80	0.81	0.79
11	Media	0.89	0.83	0.85	0.90	0.90	0.79	0.75	0.85	0.75	0.85	1	0.83	0.83	0.84	0.89	0.87	0.79
12	Travel and Leisure	0.78	0.79	0.80	0.82	0.85	0.77	0.70	0.82	0.71	0.80	0.83	1	0.69	0.70	0.78	0.81	0.80
13	Telecommunications	0.82	0.73	0.69	0.80	0.77	0.65	0.67	0.71	0.66	0.76	0.83	0.69	1	0.91	0.89	0.80	0.68
14	Utilities	0.87	0.78	0.78	0.85	0.83	0.72	0.65	0.74	0.66	0.78	0.84	0.70	0.91	1	0.91	0.84	0.74
15	Insurance	0.89	0.82	0.82	0.90	0.86	0.75	0.66	0.78	0.71	0.80	0.89	0.78	0.89	0.91	1	0.86	0.78
16	Financial Services	0.87	0.86	0.87	0.90	0.92	0.84	0.65	0.85	0.72	0.81	0.87	0.81	0.80	0.84	0.86	1	0.81
17	Technology	0.80	0.77	0.81	0.82	0.85	0.74	0.63	0.78	0.69	0.79	0.79	0.80	0.68	0.74	0.78	0.81	1