Online Appendices to: “Robust Real Rate Rules”

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Appendix A Non-linear expectational difference equations

We are interested in the non-linear expectational difference equation:

$$\left(\frac{\Pi_{t-1}^{*}}{\Pi_t}\right)^{\phi} = \mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} \frac{\Pi_t^{*}}{\Pi_{t+1}}.$$

If we define $X_t := \frac{\Pi_{t-1}^{*}}{\Pi_t}$ and $Z_t := \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}}$ then this difference equation is a particular example of the more general equation:

$$X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1}.$$

We show in Appendix A.1 that if $Z_t = 1$ for all t, then this has a unique solution for $\phi > 1$, and we show in Appendix A.2 that it still has a unique solution for arbitrary Z_t under a few additional conditions, and that the solution is approximately unique under even milder conditions.

For the results of Appendix A.2 to apply, we need that Π_t is bounded above. This is true in any model with monopolistic competition in which at least some small fraction of firms do not adjust their price each period. This does not seem an unrealistic assumption, at least if the model's time periods are sufficiently short. Even under hyper-inflation, it is still unlikely that firms adjust prices many times per day.

Π_t is bounded above in such a model because the price level remains finite even if adjusting firms set an infinite price, as all demand switches to non-adjusting firms. For example, the model of Fernández-Villaverde et al. (2015) contains the equation:

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta) \widetilde{\Pi}_t^{\,1-\varepsilon},$$

where $\widetilde{\Pi}_t$ is the relative price of adjusting firms and $\varepsilon > 1$. This equation comes from the definition of the aggregate price. As $\widetilde{\Pi}_t \to \infty$, $\Pi_t \to \theta^{-\frac{1}{\varepsilon-1}} < \infty$, thus inflation is always bounded above, as required.

A.1 Uniqueness of the solution of a simple non-linear expectational difference equation

Let $\phi > 1$. We seek to prove that the non-linear expectational difference equation:

$$X_t^{\phi} = \mathbb{E}_t X_{t+1},$$

has a unique solution that is:

a) positive (i.e., $X_t > 0$ for all $t \in \mathbb{Z}$),

b) strictly stationary (so for example $\mathbb{E}X_t = \mathbb{E}X_s$ for all $t, s, \in \mathbb{Z}$),

c) and has bounded unconditional mean and log mean (i.e., $\mathbb{E}X_t < \infty$ and $|\mathbb{E} \log X_t| < \infty$ for all $t \in \mathbb{Z}$).

Clearly $X_t = 1$ is one such solution.

Let X_t be a solution to $X_t^{\phi} = \mathbb{E}_t X_{t+1}$ satisfying (a), (b) and (c) above. Let $x_t := \log X_t$. Then from taking logs, we have:

$$\phi x_t = \log \mathbb{E}_t \exp x_{t+1} \geq \log \exp \mathbb{E}_t x_{t+1} = \mathbb{E}_t x_{t+1},$$

by Jensen's inequality. Therefore, by the law of iterated expectations, for any $k \in \mathbb{N}$:

$$\phi^k x_t \geq \mathbb{E}_t x_{t+k} = \mathbb{E}_t x_{t+k}.$$

As $k \to \infty$, the left-hand side tends to either plus infinity (if $x_t > 0$), zero (if $x_t = 0$), or minus infinity (if $x_t < 0$). On the other hand, as $k \to \infty$, the right-hand side tends to $\mathbb{E}x_t > -\infty$, by stationarity. Thus, we must have that $x_t \geq 0$ for all $t \in \mathbb{Z}$, else this equation would be violated. Hence, $X_t \geq 1$ for all $t \in \mathbb{Z}$.

Now note that by stationarity, the law of iterated expectations and Jensen's inequality:

$$\mathbb{E}X_t = \mathbb{E}X_{t+1} = \mathbb{E}\mathbb{E}_t X_{t+1} = \mathbb{E}X_t^{\phi} \geq (\mathbb{E}X_t)^{\phi},$$

so $1 \geq (\mathbb{E}X_t)^{\phi-1}$, meaning $\mathbb{E}X_t \leq 1$. However, since $X_t \geq 1$ for all $t \in \mathbb{Z}$, the only way we can have that $\mathbb{E}X_t \leq 1$ is if in fact $X_t = 1$ for all $t \in \mathbb{Z}$.

Therefore, $X_t \equiv 1$ is the unique solution to the original expectational difference equation satisfying (a), (b) and (c) above.

A.2 Uniqueness of the solution of a more general non-linear difference equation

Let $\underline{\phi} \geq 1$ and let $(Z_t)_{t\in\mathbb{Z}}$ be a stochastic process satisfying the following conditions:

i) $Z_t > 0$, for all $t \in \mathbb{Z}$,

ii) $\mathbb{E}_t Z_{t+1} = 1$, for all $t \in \mathbb{Z}$,

iii) $(Z_t)_{t\in\mathbb{Z}}$ is strictly stationary,

iv) there exists $\overline{Z} \geq 1$, independent of the stochastic process $(X_t)_{t\in\mathbb{Z}}$ (to be introduced), such that for all $\phi > \underline{\phi}$, and for all $t \in \mathbb{Z}$ and all $k \in \mathbb{N}$ with $k > 0$, $\mathbb{E}_t Z_{t+k}^{\frac{\phi}{\phi-1}} \leq \overline{Z^{\frac{\phi}{\phi-1}}}$.

The larger is $\underline{\phi}$, the weaker is the moment boundedness assumptions (iv). For example, if $\underline{\phi} = 2$, then this just requires bounded second moments.

Let $\underline{X} \in (0,1)$ and let $\phi > \underline{\phi}$. We seek to prove that the non-linear expectational difference equation:

$$X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1},$$

has a unique solution that is:

a) bounded below by $\underline{X}$ (so $X_t > \underline{X} > 0$ for all $t \in \mathbb{Z}$),

b) strictly stationary (so for example $\mathbb{E}X_t = \mathbb{E}X_s$ for all $t, s, \in \mathbb{Z}$),

c) and has bounded unconditional mean, ϕ^{th} mean and log mean (i.e., $\mathbb{E}X_t < \infty$, $\mathbb{E}X_t^{\phi} < \infty$ and $|\mathbb{E}\log X_t| < \infty$ for all $t \in \mathbb{Z}$).

Clearly $X_t = 1$ is one such solution. Note that Z_t may be a function of X_t and its history, so Z_t and X_t are not guaranteed to be independent. The previous subappendix covers the case with $Z_t \equiv 1$ in which slightly weaker assumptions are needed.

First note that for all $t \in \mathbb{Z}$:

$$\begin{aligned} 1 = \mathbb{E}_t Z_{t+1} &= \mathbb{E}_t[Z_{t+1}1] = \mathbb{E}_t\big[Z_{t+1}\mathbb{E}_{t+1}[Z_{t+2}1]\big] = \mathbb{E}_t\big[\mathbb{E}_{t+1}[Z_{t+1}Z_{t+2}1]\big] \\ &= \mathbb{E}_t[Z_{t+1}Z_{t+2}1] = \mathbb{E}_t\big[Z_{t+1}Z_{t+2}\mathbb{E}_{t+2}[Z_{t+3}1]\big] = \cdots \\ &= \mathbb{E}_t\left[\prod_{j=1}^{k} Z_{t+j}\right], \qquad \forall k \in \mathbb{N}, \end{aligned}$$

by assumption (ii) and the law of iterated expectations.

Now let $x_t := \log X_t$ and $\underline{x} := \log \underline{X}$. Then from taking logs, we have:

$$\phi x_t = \log \mathbb{E}_t Z_{t+1} \exp x_{t+1} \geq \log \exp \mathbb{E}_t Z_{t+1} x_{t+1} = \mathbb{E}_t Z_{t+1} x_{t+1},$$

by Jensen's inequality, as $\mathbb{E}_t[Z_{t+1} \times (\cdot)]$ defines a measure since $\mathbb{E}_t Z_{t+1} = 1$. Therefore, by the law of iterated expectations, for any $k \in \mathbb{N}$:

$$\phi^k x_t \geq \mathbb{E}_t \left[\prod_{j=1}^{k} Z_{t+j}\right] x_{t+k} \geq \mathbb{E}_t \left[\prod_{j=1}^{k} Z_{t+j}\right] \underline{x} = \underline{x} > -\infty,$$

by the result of the previous paragraph. As $k \to \infty$, the left-hand side tends to either plus infinity (if $x_t > 0$), zero (if $x_t = 0$), or minus infinity (if $x_t < 0$). Thus, we must have that $x_t \geq 0$ for all $t \in \mathbb{Z}$, else this equation would be violated. Hence, $X_t \geq 1$ for all $t \in \mathbb{Z}$.

Now, define $\overline{z} := \log \overline{Z}$, and for all $t \in \mathbb{Z}$ and all $k \in \mathbb{N}$ with $k > 0$ define:

$$\tilde{z}_{t,t+k} := \log \left[\mathbb{E}_t Z_{t+k}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} < \overline{z},$$

by our assumptions (iv). Then by repeatedly applying Hölder's inequality:

$$\begin{aligned}
X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1} &\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t X_{t+1}^{\phi}\right]^{\frac{1}{\phi}} \\
&\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t \left[\left[\mathbb{E}_{t+1} Z_{t+2}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_{t+1} X_{t+2}^{\phi}\right]^{\frac{1}{\phi}}\right]\right]^{\frac{1}{\phi}} \\
&\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t Z_{t+2}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi^2}} \left[\mathbb{E}_t X_{t+2}^{\phi}\right]^{\frac{1}{\phi^2}} \\
&\leq \cdots \\
&\leq \prod_{j=1}^{k} \left[\mathbb{E}_t Z_{t+j}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi^j}} \left[\mathbb{E}_t X_{t+k}^{\phi}\right]^{\frac{1}{\phi^k}},
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k > 0$. Thus, from taking logs and limits:

$$x_t \leq \sum_{j=1}^{\infty} \phi^{-j} \tilde{z}_{t,t+j} + \frac{1}{\phi} \lim_{k\to\infty} \left[\phi^{-k} \log \mathbb{E}_t X_{t+k}^{\phi}\right] = \sum_{j=1}^{\infty} \phi^{-j} \tilde{z}_{t,t+j} \leq \frac{\overline{z}}{\phi - 1},$$

where the equality follows from the fact that by stationarity, $\lim_{k\to\infty} \mathbb{E}_t X_{t+k}^{\phi} = \mathbb{E}X_t^{\phi} < \infty$. Thus, $X_t \le \overline{Z}^{\frac{1}{\phi-1}}$ for all $t \in \mathbb{Z}$. By assumption $\overline{Z}$ is not a function of ϕ, so as $\phi \to \infty$, this upper bound on X_t tends to 1. Hence, for large ϕ, $X_t \approx 1$, giving approximate uniqueness.

We can derive even stronger results in the case in which $\underline{\phi} = 1$ (in our assumptions) and one additional assumption holds. First note that with $\underline{\phi} = 1$, from taking limits as $\phi \to 1$ in assumption (iv), we must have that $Z_t \le \overline{Z}$ with probability one (for all $t \in \mathbb{Z}$).

Let Z_t^* be the value that would be taken by Z_t if it were the case that $X_t = 1$ for all $t \in \mathbb{Z}$. So, it is also the case that $Z_t^* \le \overline{Z}$ with probability one (for all $t \in \mathbb{Z}$), by our assumption (iv). Suppose further that there exists $\kappa \ge 0$ such that:

$$\mathbb{E}|Z_t - Z_t^*| \le \kappa \mathbb{E}(X_t - 1).$$

This is reasonable, since if $X_t \to 1$ (almost surely), we expect that $Z_t \to Z_t^*$ (almost surely) as well.

Now note that:

$$\mathbb{E}(X_t - 1) = \mathbb{E}\left[(\mathbb{E}_t Z_{t+1} X_{t+1})^{\frac{1}{\phi}} - 1\right] \le \mathbb{E}\left[\frac{1}{\phi}(\mathbb{E}_t Z_{t+1} X_{t+1} - 1)\right] = \frac{1}{\phi}[\mathbb{E}Z_t X_t - 1],$$

(using stationarity and the law of iterated expectations in the final equality). Thus:

$$\begin{aligned}
\mathbb{E}(X_t - 1) &= \mathbb{E}\left[(\mathbb{E}_t Z_{t+1} X_{t+1})^{\frac{1}{\phi}} - 1\right] \le \mathbb{E}\left[\frac{1}{\phi}(\mathbb{E}_t Z_{t+1} X_{t+1} - 1)\right] = \frac{1}{\phi}[\mathbb{E}Z_t X_t - 1] \\
&= \frac{1}{\phi}[\mathbb{E}Z_t X_t - \mathbb{E}Z_t^*] = \frac{1}{\phi}[\mathbb{E}(Z_t - Z_t^*)X_t + \mathbb{E}Z_t^*(X_t - 1)] \\
&\le \frac{1}{\phi}[\mathbb{E}|Z_t - Z_t^*|X_t + \mathbb{E}Z_t^*(X_t - 1)] \le \frac{1}{\phi}\left[\kappa \mathbb{E}(X_t - 1)\overline{Z}^{\frac{1}{\phi-1}} + \overline{Z}\mathbb{E}(X_t - 1)\right] \\
&= \frac{1}{\phi}\left[\kappa \overline{Z}^{\frac{1}{\phi-1}} + \overline{Z}\right]\mathbb{E}(X_t - 1),
\end{aligned}$$

(from, respectively, the convexity of $y \mapsto y^{\frac{1}{\phi}}$, stationarity and the law of iterated expectations, the fact that $\mathbb{E}Z_t^* = 1$, algebra, that $y \le |y|$, our bounds on X_t, $\mathbb{E}|Z_t - Z_t^*|$ and Z_t^*, and more algebra). As $\phi \to \infty$, $\kappa \overline{Z}^{\frac{1}{\phi-1}} + \overline{Z} \to \kappa + \overline{Z} < \infty$, so for large ϕ it must be the case that $\frac{1}{\phi}\left[\kappa \overline{Z}^{\frac{1}{\phi-1}} + \overline{Z}\right] < 1$. Hence if ϕ is large enough for this to hold, then

$\mathbb{E}(X_t - 1) \leq 0$. However, since $X_t \geq 1$ for all $t \in \mathbb{Z}$, the only way we can have that $\mathbb{E}X_t \leq 1$ is if in fact $X_t = 1$ for all $t \in \mathbb{Z}$.

Therefore, for large enough ϕ, $X_t \equiv 1$ is the unique solution to the original expectational difference equation satisfying (a), (b) and (c) above.

Appendix B Fiscal Theory of the Price Level (FTPL) results

B.1 Exact equilibria under active fiscal policy with geometric coupon debt and flexible prices

Suppose the representative household supplies one unit of labour, inelastically. Production of the final good is given by:

$$y_t = l_t (= 1).$$

In period 0, the representative household maximises:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to the budget constraint:

$$P_t c_t + A_t + Q_t B_t + P_t \tau_t = P_t y_t + I_{t-1} A_{t-1} + B_{t-1}(1 + \omega Q_t),$$

where c_t is consumption, τ_t are real lump sum taxes, P_t is the price of the final good, A_t is the number of one period nominal bonds purchased by the household at t, which each return I_t in period $t+1$, Q_t is the price of a long (geometric coupon) bond and B_t are the number of units of this long bond purchased by the household at t. One unit of the period t long bond bought at t returns \$1 at $t+1$, along with ω units of the period $t+1$ bond.

The household first order conditions imply:

$$1 = \beta I_t \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}},$$

$$Q_t = \beta \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}} (1 + \omega Q_{t+1}).$$

The household transversality conditions are that:

$$\lim_{t\to\infty} \beta^t \frac{A_t}{P_t c_t} = 0,$$

$$\lim_{t\to\infty} \beta^t \frac{Q_t B_t}{P_t c_t} = 0.$$

The government fixes taxes at a constant positive level:

$$\tau_t = \tau, \qquad \tau > 0.$$

The government issues no one period bonds, so:

$$A_t = 0.$$

The central bank pegs nominal interest rates at:

$$I_t = \beta^{-1}.$$

(We will discuss active monetary policy later.)

The final goods market clears, so:

$$y_t = c_t = 1.$$

Thus, from the household budget constraint, we have the following government budget constraint:

$$Q_t B_t + P_t \tau = B_{t-1}(1 + \omega Q_t).$$

We look for an equilibrium in which $P_t = P$ for all $t \geq 0$. We do not impose a priori that $P = P_{-1}$.

With $P_t = P$ for $t \geq 0$, the household Euler equations simplify to (respectively):

$$1 = \beta I_t,$$

$$Q_t = \beta \mathbb{E}_t (1 + \omega Q_{t+1}).$$

The former equation is consistent with the CB's peg of $I_t = \beta^{-1}$.

We consider the following solution to the latter equation:

$$Q_t = \frac{\beta}{1-\beta\omega} + \left(Q_0 - \frac{\beta}{1-\beta\omega}\right)(\beta\omega)^{-t}.$$

We wish to find Q_0, which is free to jump. There are three cases to consider:

Case 1: $Q_0 < \frac{\beta}{1-\beta\omega}$. Then Q_t eventually goes to zero (and then negative), which certainly cannot be consistent with a world in which $I_t > 0$. Thus, this case is ruled out.

Case 2: $Q_0 = \frac{\beta}{1-\beta\omega}$. Then Q_t is constant, and the government budget constraint becomes:

$$B_t = \beta^{-1}B_{t-1} - \beta^{-1}(1-\beta\omega)P\tau.$$

Thus:

$$B_t = P\tau\frac{1-\beta\omega}{1-\beta} + \left(B_{-1} - P\tau\frac{1-\beta\omega}{1-\beta}\right)\beta^{-t-1}$$

So:

$$\begin{aligned}\beta^t\frac{Q_tB_t}{P_tc_t} &= \frac{\beta}{1-\beta\omega}\frac{1}{P}\left[P\tau\frac{1-\beta\omega}{1-\beta}\beta^t + \left(B_{-1} - P\tau\frac{1-\beta\omega}{1-\beta}\right)\beta^{-1}\right] \\ &\to \frac{1}{1-\beta\omega}\frac{1}{P}\left(B_{-1} - P\tau\frac{1-\beta\omega}{1-\beta}\right)\end{aligned}$$

as $t \to \infty$.

Thus, from the transversality constraint:

$$P = \frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}.$$

This is the standard FTPL equilibrium. **Equilibrium type 1!**

Case 3: $Q_0 > \frac{\beta}{1-\beta\omega}$.

Define:

$$q_t := Q_t(\beta\omega)^t,$$

$$b_t := B_t\omega^{-t}.$$

Then the government budget constraint states:

$$b_t = \left(1 + \frac{(\beta\omega)^t}{\omega q_t}\right)b_{t-1} - \frac{\beta^t P\tau}{q_t},$$

and the transversality constraint states:

$$\frac{1}{P}\lim_{t\to\infty} q_t b_t = 0.$$

By our solution for q_t, we know that $q_t \to Q_0 - \frac{\beta}{1-\beta\omega} > 0$. Thus, the transversality condition requires:

$$\lim_{t\to\infty} b_t = 0.$$

Now define:

$$\hat{b}_t := \frac{b_t}{\prod_{k=0}^{t}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right)},$$

with $\hat{b}_{-1} = b_{-1} = \omega B_{-1}$.

The denominator in the definition of $\hat{b}_t$ is greater than 1, so if $b_t \to 0$ as $t \to \infty$, then certainly $\hat{b}_t \to 0$. Likewise, if $\hat{b}_t \to 0$ as $t \to \infty$, then also $b_t \to 0$, since for all t:

$$\begin{aligned}
&\prod_{k=0}^{t}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right) \le \prod_{k=0}^{\infty}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right) \\
&= \prod_{k=0}^{\infty}\left(1 + \frac{1-\beta\omega}{\beta\omega + \omega\big((1-\beta\omega)Q_0 - \beta\big)(\beta\omega)^{-k}}\right) \\
&= \exp\sum_{k=0}^{\infty}\log\left(1 + \frac{1-\beta\omega}{\beta\omega + \omega\big((1-\beta\omega)Q_0 - \beta\big)(\beta\omega)^{-k}}\right) \\
&\le \exp\int_{-1}^{\infty}\log\left(1 + \frac{1-\beta\omega}{\beta\omega + \omega\big((1-\beta\omega)Q_0 - \beta\big)(\beta\omega)^{-k}}\right) \\
&= \frac{(1+\omega Q_0)(1-\beta\omega)}{\omega\big((1-\beta\omega)Q_0 - \beta\big)}\exp\left[\frac{1}{\log(\beta\omega)}\left[\operatorname{dilog}\left(\frac{1+\beta\omega(1+\omega Q_0)}{\beta\omega(1+\omega Q_0)}\right) + \operatorname{dilog}(\beta\omega)\right.\right. \\
&\qquad\qquad \left.\left. - \operatorname{dilog}\left(\frac{1+\beta\omega(1+\omega Q_0)}{1+\omega Q_0}\right)\right]\right] \\
&< \infty,
\end{aligned}$$

where $\operatorname{dilog}(x) := \int_1^x \frac{\log(z)}{1-z}dz$ for all x is the dilogarithm function.

Now, substituting the definition of $\hat{b}_t$ into the law of motion for b_t gives:

$$\hat{b}_t = \hat{b}_{t-1} - \frac{\beta^t P\tau}{q_t\prod_{k=0}^{t}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right)},$$

so:

$$\begin{aligned}
\hat{b}_t &= \hat{b}_{-1} - P\tau\sum_{j=0}^{t}\frac{\beta^j}{q_j\prod_{k=0}^{j}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right)} \\
&= \hat{b}_{-1} - P\tau\sum_{j=0}^{t}\frac{\prod_{k=0}^{j}\beta\left(1 + \frac{1-\beta\omega}{\beta\omega + (\omega(1-\beta\omega)Q_0 - \beta\omega)(\beta\omega)^{-k}}\right)^{-1}}{\beta\left[\frac{\beta}{1-\beta\omega}(\beta\omega)^j + \left(Q_0 - \frac{\beta}{1-\beta\omega}\right)\right]}.
\end{aligned}$$

Note that for $k \ge 0$:

$$1 < 1 + \frac{1-\beta\omega}{\beta\omega + (\omega(1-\beta\omega)Q_0 - \beta\omega)(\beta\omega)^{-k}} \le 1 + \frac{1}{\omega Q_0} < \frac{1}{\beta\omega},$$

so:

$$(\beta^2\omega)^{j+1} < \prod_{k=0}^{j} \beta \left(1 + \frac{1-\beta\omega}{\beta\omega + (\omega(1-\beta\omega)Q_0 - \beta\omega)(\beta\omega)^{-k}}\right)^{-1} < \beta^{j+1}.$$

Thus, since the denominator within the sum is converging to $\beta\big(Q_0 - \frac{\beta}{1-\beta\omega}\big)$ the sum is finite and has a finite limit as $t \to \infty$.

Hence, one equilibrium is for $Q_0 > \frac{\beta}{1-\beta\omega}$ to be arbitrary and for P to be given by:

$$P = \frac{\hat{b}_{-1}}{\tau \sum_{j=0}^{\infty} \frac{\prod_{k=0}^{j} \beta \left(1 + \frac{1-\beta\omega}{\beta\omega + (\omega(1-\beta\omega)Q_0 - \beta\omega)(\beta\omega)^{-k}}\right)^{-1}}{\beta\left[\frac{\beta}{1-\beta\omega}(\beta\omega)^j + \left(Q_0 - \frac{\beta}{1-\beta\omega}\right)\right]}}.$$

Equilibrium type 2!

Alternatively, suppose P is given. When can we solve the previous equation to find Q_0? As $Q_0 \to \frac{\beta}{1-\beta\omega}$, the right-hand side of the previous equation tends to:

$$\frac{\hat{b}_{-1}}{\tau\omega}\frac{1-\beta}{1-\beta\omega} = \frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}.$$

As $Q_0 \to \infty$, this right-hand side tends to ∞. Thus, by the intermediate value theorem, for any $P \in \left[\frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}, \infty\right)$, there is a Q_0 that satisfies the transversality constraint. Hence, inflation is unbounded about in the initial period.

Equilibrium type 3!

Therefore, the FTPL implies a lower bound on the price level, not an upper bound, and so with passive monetary policy, there are multiple equilibria.

Now suppose that monetary policy is active, with:

$$I_t = \beta^{-1}\Pi_t^{\phi},$$

with $\phi > 1$ and $\Pi_t := \frac{P_t}{P_{t-1}}$. β^{-1} is the real interest rate in this model, so this is a nonlinear real rate rule. Given that $c_t = 1$, the Euler equation for one period bonds implies the nonlinear Fisher equation:

$$1 = \beta I_t \mathbb{E}_t \frac{1}{\Pi_{t+1}},$$

so, for $t \geq 0$:

$$\mathbb{E}_t \frac{1}{\Pi_{t+1}} = \left(\frac{1}{\Pi_t}\right)^{\phi}.$$

$\Pi_t = 1$ is the unique stationary solution to this equation, by the results of Appendix A.1 (with $X_t := \frac{1}{\Pi_t}$). In this candidate equilibrium, $I_t = \beta^{-1}$, so Π_t and I_t have the same time series as under the passive policy in the special case in which $P = P_{-1}$. Consequently, if $P_{-1} \geq \frac{B_{-1}}{\tau} \frac{1-\beta}{1-\beta\omega}$ then by the above results, there exists a Q_0 under which all equilibrium conditions and transversality conditions are satisfied. Thus, even with active monetary and active fiscal policy, there is still a stable equilibrium for inflation and real variables.

B.2 Linearised equilibria under active fiscal policy with geometric coupon debt and sticky prices

We now examine the fiscal theory of the price level in a richer model with sticky prices. We just give the linearised equations of the model. These follow equations 5.17 to 5.21 of Cochrane (2022), and the reader is referred there for the derivations. All shocks (variables of the form $\varepsilon_{\cdot,t}$) are assumed to be mean zero and independent, both across time and across shocks. The equations follow:

Euler:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma r_t.$$

Phillips:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t.$$

Fisher:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}.$$

Robust real rate rule:

$$i_t = r_t + \phi \pi_t + \varepsilon_{i,t}.$$

Exogenous real government surplus:

$$s_t = \varepsilon_{s,t}.$$

Debt evolution (v_t is the value of debt to GDP, e_t is the ex-post nominal return on government debt):

$$\beta v_t = v_{t-1} + e_t - \pi_t - s_t.$$

Equal returns:

$$\mathbb{E}_t e_{t+1} = i_t.$$

Bond pricing (ω controls the maturity structure. $\omega = 0$ is one period debt, $\omega = 1$ is a perpetuity):

$$e_t = \omega q_t - q_{t-1}.$$

We assume that $\omega > 0$. Then for any $\phi \neq 0$, the following solves these linear expectational difference equations:

$$\pi_t = -\frac{\varepsilon_{i,t}}{\phi}, \quad x_t = -\frac{\varepsilon_{i,t}}{\kappa\phi},$$
$$r_t = \frac{\varepsilon_{i,t}}{\sigma\kappa\phi}, \quad v_t = -\frac{\varepsilon_{i,t}}{\sigma\kappa\phi},$$
$$e_t = \varepsilon_{s,t} - \left(\frac{\beta}{\sigma\kappa\phi} + \frac{1}{\phi}\right)\varepsilon_{i,t} + \frac{\varepsilon_{i,t-1}}{\sigma\kappa\phi},$$
$$q_t = \frac{1}{\omega}\left[q_{t-1} + \varepsilon_{s,t} - \left(\frac{\beta}{\sigma\kappa\phi} + \frac{1}{\phi}\right)\varepsilon_{i,t} + \frac{\varepsilon_{i,t-1}}{\sigma\kappa\phi}\right].$$

As in the non-linear, flexible price case, the bond price is exploding. However, the real value of government debt remains stationary, which is sufficient for the transversality constraint to be satisfied. Inflation and all real variables are also stationary. Thus, if monetary policy is passive ($\phi \in (0,1)$), then the linearised model has multiple valid equilibria, this one, and the standard "FTPL" one in which q_t is stationary (see Cochrane (2022)). Conversely, if monetary policy is active ($\phi > 1$), then the model possesses a valid equilibrium with stationary inflation and real variables.

B.3 Stability under real rate rules for generic models

When is the real rate rule $i_t = r_t + \phi\pi_t + \varepsilon_{\zeta,t}$ with $\phi > 1$ consistent with stable real variables?

We need to impose at least some additional structure on the rest of the model in order to make progress on this question for general models. In particular, we assume that the other endogenous variables of the model can be partitioned into two groups, z_t and q_t, where z_t may affect q_t but not vice versa. The variables in z_t must be stationary in equilibrium, but always have a unique stationary solution if π_t is stationary. The variables in q_t need not be stationary in equilibrium. These restrictions are satisfied by models of the fiscal theory of the price level, for example, in which case hours, output, consumption, investment, debt-to-GDP, inflation, nominal & real rates and so on will be in z_t, while bond prices and quantities will be in q_t. That bond prices and quantities need not be stationary under the fiscal theory of the price level was carefully established from transversality conditions in Appendix B.1, under the assumption of geometric coupon debt. The calculations of Appendices B.1 and B.2 also show that only the value of government debt matters for "z_t" variables, not its decomposition into bond prices and quantities.

Then, without loss of generality, the linearized model (without the monetary rule) must have a representation in the following form:[1]

$$0 = A_{zz}\mathbb{E}_t z_{t+1} + B_{zz} z_t + C_{zz} z_{t-1} + d_z \pi_t + E_z \nu_t, \tag{9}$$

$$0 = A_{qq}\mathbb{E}_t q_{t+1} + B_{qq} q_t + C_{qq} q_{t-1} + A_{qz}\mathbb{E}_t z_{t+1} + B_{qz} z_t + C_{qz} z_{t-1} + d_q \pi_t + E_q \nu_t, \tag{10}$$

where ν_t is a vector of exogenous shocks with $\mathbb{E}_{t-1}\nu_t = 0$, and where the coefficient matrices are such that there is a unique matrix F_z with eigenvalues in the unit circle such that $F_z = -(A_{zz}F_z + B_{zz})^{-1}C_{zz}$. This condition on F_z imposes that z_t has a stationary solution if π_t is stationary: in other words, it ensures there is no real indeterminacy in the model. Note that q_t (and its lags and leads) do not enter the equation for z_t, by our assumption that q_t does not affect z_t.

We want to see if $\pi_t = -\frac{1}{\phi}\varepsilon_{\zeta,t}$ is consistent with (9) and (10). This is the only possible stationary solution for inflation under the real rate rule $i_t = r_t + \phi\pi_t + \varepsilon_{\zeta,t}$ with $\phi > 1$. From this solution for π_t, (9) and the definition of F_z:

[1] The lack of terms in $\mathbb{E}_t\pi_{t+1}$ and π_{t-1} is without loss of generality, as such responses can be included by adding an auxiliary variable $z_{t,j}$ with an equation of the form $z_{t,j} = \pi_t$.

$$z_t = F_z z_{t-1} + (A_{zz}F_z + B_{zz})^{-1}\left(\frac{1}{\phi}d_z\varepsilon_{\zeta,t} - E_z\nu_t\right).$$

Hence, from (10):

$$\begin{aligned}0 = A_{qq}\mathbb{E}_t q_{t+1} + B_{qq}q_t + C_{qq}q_{t-1} + \Big(\big(A_{qz}F_z + B_{qz}\big)F_z + C_{qz}\Big)z_{t-1} \\ + \big(A_{qz}F_z + B_{qz}\big)(A_{zz}F_z + B_{zz})^{-1}\left(\frac{1}{\phi}d_z\varepsilon_{\zeta,t} - E_z\nu_t\right) - \frac{1}{\phi}d_q\varepsilon_{\zeta,t} + E_q\nu_t.\end{aligned}$$

If there is a real matrix F_q solving $F_q = -\big(A_{qq}F_q + B_{qq}\big)^{-1}C_{qq}$ then q_t admits a solution of the form:

$$q_t = F_q q_{t-1} + G z_{t-1} + h\varepsilon_{\zeta,t} + J\nu_t,$$

for some matrices G and J and some vector h. This may be explosive, but that is allowed by our assumptions. (In the fiscal theory of the price level contexts, this corresponds to explosions in bond prices and quantities of opposite signs, producing stable debt values.) In this case, there is no inconsistency with the solution for inflation implied by our real rate rule. So, the answer to the question "is a real rate rule consistent with stable z_t variables?" is the same as the answer to the question "does $A_{qq}F_q^2 + B_{qq}F_q + C_{qq}$ have a real solution for F_q?".

When $A_{qq} = 0$, this is simple. A real solution exists if and only if B_{qq} is full rank. Generically, matrices are full rank, so except in knife edge cases, a real solution exists when $A_{qq} = 0$. Furthermore, by continuity, for almost all A_{qq} with sufficiently small norm, a real solution must exist. Under standard models of the fiscal theory of the price level, $A_{qq} = 0$, since the geometric coupon bond first order condition $Q_t = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}}(1 + \omega Q_{t+1})$ can be rewritten as the two equations $E_t = \frac{1+\omega Q_t}{Q_{t-1}}$, and $1 = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} E_{t+1}$ (E_t is in z_t, while Q_t is in q_t, also see Appendices B.1 and B.2). Thus, generically, all models sufficiently close to a standard fiscal theory of the price level model must have a real solution for F_q. Therefore, for all such models, a real rate rule is consistent with a stationary path for z_t variables.

Appendix C Welfare in New Keynesian models

In Subsection 1.4, we established that real rate rules like equation (7) could exactly mimic any other time invariant policy, with the time varying inflation target only responding to structural shocks and their lags. Thus, real rate rules can mimic unconditionally optimal policy, optimal commitment policy from a timeless perspective, or optimal discretionary policy. Hence, these rules can achieve high welfare.

We begin this section by looking at unconditionally optimal time-invariant policy using real rate rules, in a simple NK model. We then go on to analyse the performance of these rules if further restrictions are placed upon them, such as only permitting the central bank to respond to current or sufficiently recent shocks. We show that optimal policy in estimated models of the US economy comes close to stabilizing inflation, with optimal inflation dynamics describable by an ARMA process with few MA terms.

C.1 Welfare in the basic three equation model

Any welfare analysis requires us to specify the rest of the model, as welfare is generally a function of output's variability, not just that of inflation. Thus, as a first example suppose that inflation and output are linked by the standard Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t,$$

where x_t is the output gap, and ω_t is a mark-up shock, which is assumed IID with mean zero. Additionally, suppose that the policy maker wants to minimise the unconditional expectation of a quadratic loss function in inflation and the output gap. I.e., the period t policy maker minimises:

$$(1-\beta)\mathbb{E} \sum_{k=0}^{\infty} \beta^k \left(\pi_{t+k}^2 + \lambda x_{t+k}^2\right),$$

for some $\lambda > 0$ and $\beta \in (0,1)$.

We suppose that the policy maker is constrained to choose a time-invariant (i.e., stationary) policy, thus the objective simplifies to:[2]

$$\mathbb{E}\left(\pi_t^2 + \lambda x_t^2\right).$$

As the policy maker only cares about inflation and output gaps, with the former being effectively under their control, and the latter only determined by inflation and mark-up shocks, the optimal policy must have the form:

$$\pi_t = \kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k},$$

for some $\theta_0, \theta_1, \dots$ to be determined. We have already shown that such a policy may be determinately implemented via a rule of the form of (7).

Substituting this policy into the Phillips curve then gives:

$$\sum_{k=0}^{\infty} \theta_k \omega_{t-k} = \beta \sum_{k=0}^{\infty} \theta_{k+1} \omega_{t-k} + x_t + \omega_t,$$

so:

$$x_t = \sum_{k=0}^{\infty} (\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k}.$$

Hence, the policy maker's objective is to choose $\theta_0, \theta_1, \dots$ to minimise:

$$\mathbb{E}\left(\pi_t^2 + \lambda x_t^2\right) = \mathbb{E}[\omega_t^2] \sum_{k=0}^{\infty} \left[\kappa^2 \theta_k^2 + \lambda(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])^2\right].$$

The first order conditions then give:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\forall k > 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \beta\frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

The solution of these conditions is given in Appendix E.13. Unsurprisingly, this agrees with the unconditionally optimal solution given in the prior literature (e.g. Damjanovic, Damjanovic & Nolan (2008)), which satisfies:

$$\pi_t + \frac{\lambda}{\kappa}(x_t - \beta x_{t-1}) = 0,$$

[2] See e.g. Damjanovic, Damjanovic & Nolan (2008).

i.e.:

$$\kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k} + \frac{\lambda}{\kappa}\left[\sum_{k=0}^{\infty}(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} \right. \\ \left. - \beta \sum_{k=1}^{\infty}(\theta_{k-1} - \beta\theta_k - \mathbb{1}[k-1=0])\omega_{t-k}\right] = 0.$$

To see the equivalence, note that from matching coefficients, this equation holds if and only if the above first order conditions hold. We will present a convenient representation of the solution to these equations below.

Additionally, note that as $\frac{\lambda}{\kappa^2} \to 0$, $\theta_k \to 0$ for all $k \in \mathbb{N}$. In other words, if the central bank does not care about the output gap, then they optimally choose to have constant inflation, i.e., to follow the rule from equation (2). The central bank also chooses constant inflation if the Phillips curve is vertical (i.e., $\kappa = \pm\infty$). In this case, neither inflation nor mark-up shocks have any impact on the output gap.

C.2 Optimal policy under limited central bank memory

The first order conditions derived above also enable us to easily solve for optimal unconditional policy under limited memory. For example, if the central bank does not "remember" $\omega_{t-1}, \omega_{t-2}, \ldots$, so uses a rule that is only a function of ω_t at t, then the optimal θ_0 will satisfy the above first order conditions with $\theta_1 = \theta_2 = \cdots = 0$. This means:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - 1) = 0,$$

so $\theta_0 = \frac{\lambda}{\lambda+\kappa^2}$. It turns out that this exactly coincides with the solution under discretion (see Appendix E.14).

If the central bank can "remember" ω_{t-1}, so π_t is an MA(1), then the optimal solution will have:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\theta_1 + \frac{\lambda}{\kappa^2}\theta_1 - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0.$$

The solution has $\theta_0 \geq 0$ and $\theta_1 \leq 0$. Thus, the shock increases π_t while reducing $\mathbb{E}_t \pi_{t+1}$, thus dampening the required movement in x_t, from the Phillips curve. We will see that this is already enough to come close to the fully optimal policy.

Going one step further, if the central bank can also "remember" π_{t-1}, then they can choose interest rates to ensure π_t follows the ARMA(1,1) process:

$$\pi_t = \rho \pi_{t-1} + \kappa \theta_0 \omega_t + \kappa \theta_1 \omega_{t-1},$$

for some ρ, θ_0, θ_1 to be determined.[3] Since US inflation appears to be well approximated by an ARMA(1,1) (Stock & Watson 2009), this may be a reasonable model of Fed behaviour. This ARMA(1,1) process has the MA(∞) representation:

$$\pi_t = \kappa \theta_0 \sum_{k=0}^{\infty} \rho^k \omega_{t-k} + \kappa \theta_1 \sum_{k=0}^{\infty} \rho^k \omega_{t-1-k} = \kappa \theta_0 \omega_t + \kappa (\rho \theta_0 + \theta_1) \sum_{k=1}^{\infty} \rho^{k-1} \omega_{t-k}. \tag{11}$$

Substituting this policy into the Phillips curve gives:

$$\theta_0 \omega_t + (\rho \theta_0 + \theta_1) \sum_{k=1}^{\infty} \rho^{k-1} \omega_{t-k} = \beta (\rho \theta_0 + \theta_1) \omega_t + \beta (\rho \theta_0 + \theta_1) \sum_{k=1}^{\infty} \rho^k \omega_{t-k} + x_t + \omega_t,$$

meaning:

$$x_t = [(1 - \beta \rho) \theta_0 - \beta \theta_1 - 1] \omega_t + (1 - \beta \rho)(\rho \theta_0 + \theta_1) \sum_{k=1}^{\infty} \rho^{k-1} \omega_{t-k}.$$

Hence, the policy maker's objective is to choose ρ, θ_0, θ_1 to minimise:

$$\mathbb{E}(\pi_t^2 + \lambda x_t^2) = \mathbb{E}[\omega_t^2] \Bigg[\kappa^2 \theta_0^2 + \lambda [(1 - \beta \rho) \theta_0 - \beta \theta_1 - 1]^2 + [\kappa^2 (\rho \theta_0 + \theta_1)^2 + \lambda (1 - \beta \rho)^2 (\rho \theta_0 + \theta_1)^2] \frac{1}{1 - \rho^2} \Bigg].$$

Tedious algebra gives that the first order conditions have solution:[4]

$$\rho = \frac{\kappa^2 + (1 + \beta^2)\lambda - \sqrt{(\kappa^2 + (1 - \beta)^2 \lambda)(\kappa^2 + (1 + \beta)^2 \lambda)}}{2 \beta \lambda}, \qquad \theta_0 = \frac{\rho}{\beta}, \qquad \theta_1 = -\rho.$$

---

[3] The targeted inflation can respond to lagged targeted inflation without changing the determinacy properties of realised inflation (always equal to targeted inflation in equilibrium). Targeted inflation cannot respond to other endogenous variables without potentially changing these determinacy properties.

[4] There is an additional solution to the first order condition with $\rho = \frac{\kappa^2 + (1+\beta^2)\lambda + \sqrt{(\kappa^2 + (1-\beta)^2\lambda)(\kappa^2 + (1+\beta)^2\lambda)}}{2\beta\lambda}$, but this is outside of the unit circle as: $\frac{\kappa^2 + (1+\beta^2)\lambda + \sqrt{(\kappa^2 + (1-\beta)^2\lambda)(\kappa^2 + (1+\beta)^2\lambda)}}{2\beta\lambda} > \frac{\kappa^2 + (1+\beta^2)\lambda + \sqrt{(\kappa^2 + (1-\beta)^2\lambda)(\kappa^2 + (1-\beta)^2\lambda)}}{2\beta\lambda} = \frac{\kappa^2 + (1-\beta+\beta^2)\lambda}{\beta\lambda} > \frac{1-\beta+\beta^2}{\beta} = \frac{1}{\beta} + \beta - 1 > 1$. However, the given solution is inside the unit circle as

As $\lambda \to 0$, or $\kappa \to \infty$, $\rho \to 0$. As $\lambda \to \infty$, or $\kappa \to 0$, $\rho \to \beta$. Since there is no other solution for κ to the equation $\rho = \beta$ than $\kappa = 0$, we must have $\rho \leq \beta$, so $\rho\theta_0 + \theta_1 \leq 0$, meaning that the response of inflation to a positive mark-up shock is again negative after the first period. Since we have one extra degree of freedom, this must attain even higher welfare than the MA(1) solution. In fact, it attains the unconditionally optimal solution. Examination of the unconditionally optimal solution from Appendix E.13 reveals that it has the same form as equation (11), thus by a revealed preference argument, the two solutions must coincide. (For example, the solution for ρ agrees with the geometric decay rate of the MA coefficients at lags beyond the first of the fully optimal solution we found in Appendix E.13.)

Hence, in a world in which the only inefficient shocks are IID cost-push shocks, the central bank can attain the unconditionally optimal welfare by ensuring inflation follows an appropriate ARMA(1,1) process. This process will have an MA coefficient equal to $-\beta \approx -0.99$, and as long as the central bank cares about output stabilisation, it will have a high degree of persistence. This is very close to the IMA(1,1) processes estimated by Dotsey, Fujita & Stark (2018) for the post-1984 period.

To see the welfare attained by the other policies we have discussed, Figure 1 plots the policy frontiers attained by varying λ for each of the polices. In all cases, we follow Eggertsson & Woodford (2003) in setting $\beta = 0.99$ and $\kappa = 0.02$. The figure makes clear that the MA(1) policy (green) is a substantial improvement on the MA(0) (discretionary) policy (red). It also shows just how close Woodford's timeless perspective (1999) (blue, hidden behind purple, derived in Appendix E.15) comes to the unconditionally optimal policy.

$\frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} > \frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1+\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} = -1,$
$\frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} < \frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1-\beta)^2\lambda)}}{2\beta\lambda} = 1.$

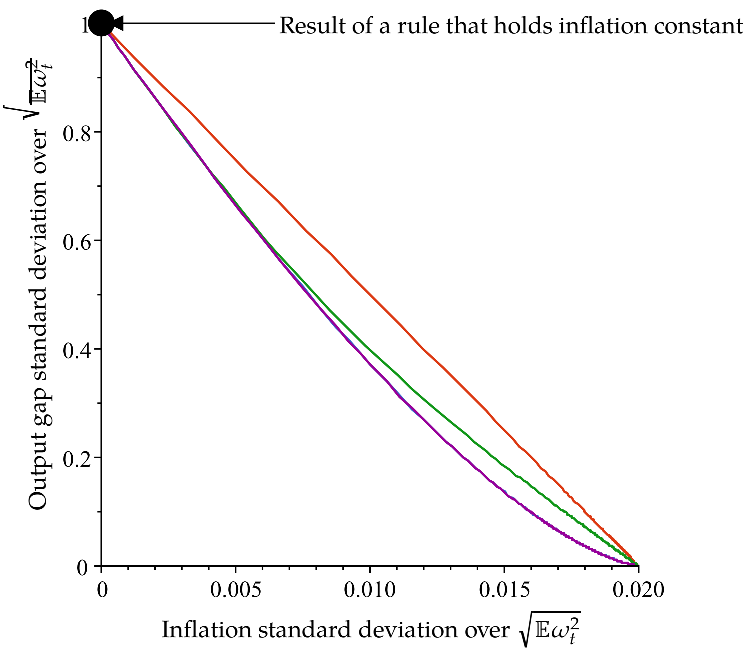

Figure 1: Policy frontiers (values attained by varying λ). $\beta = 0.99$, $\kappa = 0.02$.

Purple: Unconditionally optimal policy, equivalent to ARMA$(\mathbf{1}, \mathbf{1})$ policy.

Blue (hidden behind purple): Timeless optimal solution.

Red: Policy just responding to current shocks, equivalent to discretion.

Green: Policy that responds to current and once lagged shocks.

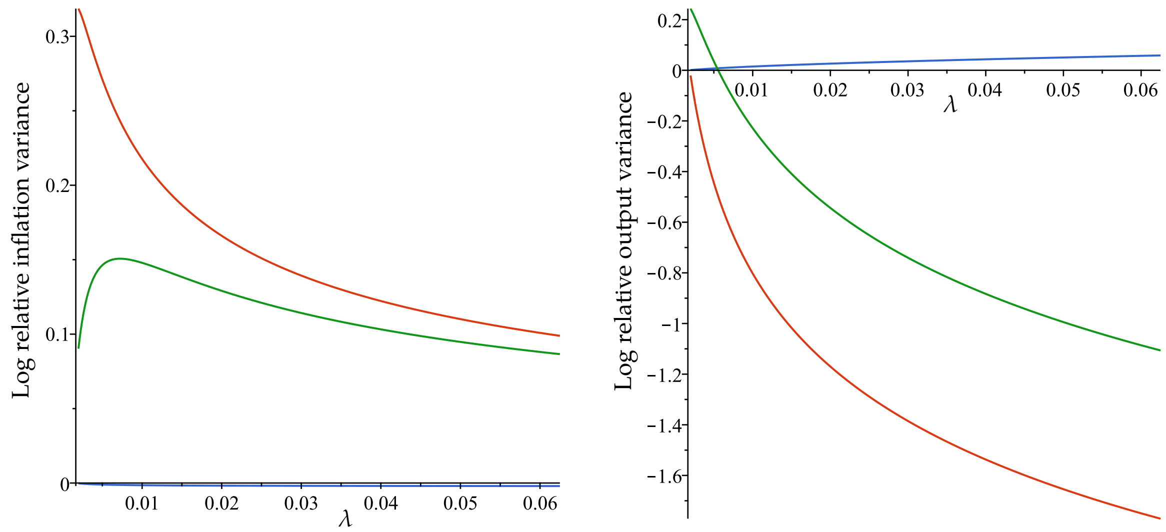

Figure 2: Logarithms of ratios of variance under a given policy to variance under unconditionally optimal policy. $\beta = 0.99$, $\kappa = 0.02$.

Blue: Timeless optimal solution.

Red: Policy just responding to current shocks, equivalent to discretion.

Green: Policy that responds to current and once lagged shocks.

Figure 2 shows how these differences across policies are driven by λ, by plotting the logarithm of the ratio of variance under a given policy to the variance under unconditionally optimal policy. We allow λ to vary from 0.002 (the value obtained by a second order approximation to the consumer's utility with $\kappa = 0.02$, if the elasticity of substitution across goods equals 10) to $\frac{1}{16}$ (corresponding to an equal weight on annual inflation and the output gap). Both the MA(0) and the MA(1) policy generate too much inflation variance and too little variance in output, relative to the unconditionally optimal solution. However, if the central bank can feasibly respond to ω_t and ω_{t-1} they can probably also respond to π_{t-1}, which is enough to deliver the unconditional optimum.

C.3 Welfare in larger NK models

Even in larger models, optimal inflation dynamics appear to be well approximated by an ARMA process with relatively few MA terms. Figure 3 shows the dynamics of observed and optimal inflation in the Justiniano, Primiceri & Tambalotti (2013) model. (This is a medium-scale New Keynesian DSGE model broadly similar to the model of Smets & Wouters (2007).) While actual inflation is highly persistent, with the same shocks hitting the economy, optimal inflation is far less persistent, with the sample autocorrelation essentially insignificant at 95% after four lags.

Note that for any $\rho \in (-1,1)$, the solution for optimal inflation has a multiple shock, ARMA$(1,\infty)$ representation of the form:

$$\pi_t - \pi = \rho(\pi_{t-1} - \pi) + \sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k},$$

where $\varepsilon_{1,t}, \dots, \varepsilon_{N,t}$ are the model's structural shocks. We can approximate this process by truncating the MA terms at some point, e.g., by considering the multiple shock ARMA$(1,K)$ process:

$$\pi_t^{(K)} - \pi = \rho\big(\pi_{t-1}^{(K)} - \pi\big) + \sum_{k=0}^{K} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k}.$$

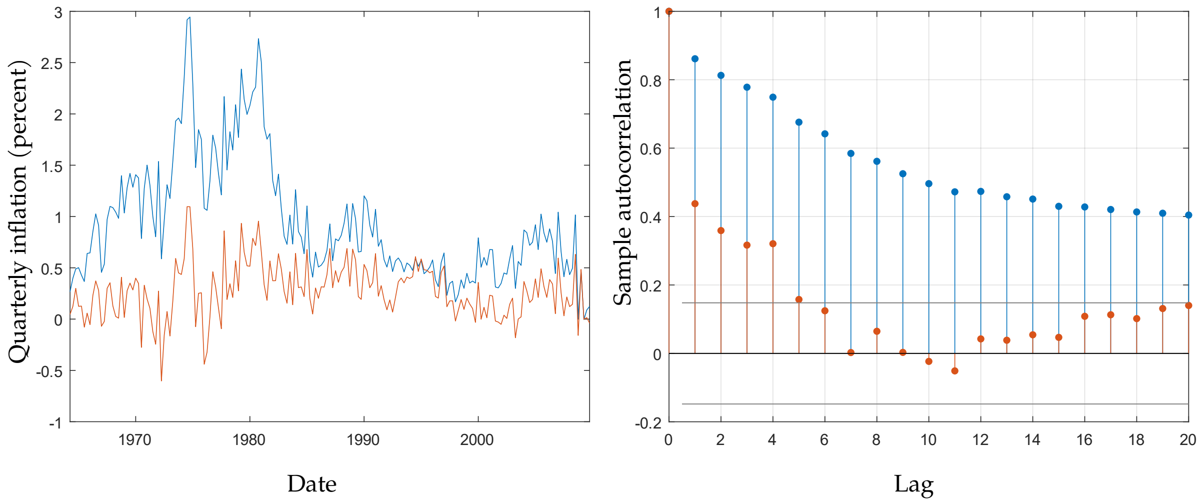

Figure 3: Behaviour of realised inflation (blue) and optimal inflation (red) in the Justiniano, Primiceri & Tambalotti (2013) model.

Left panel shows the timeseries. Right panel shows their sample autocorrelation.

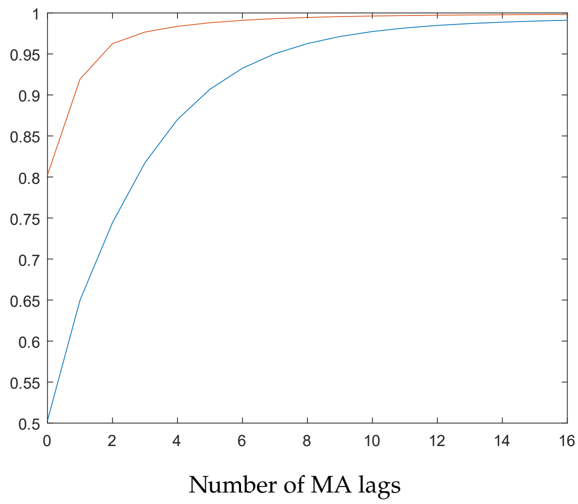

Figure 4: Proportion of the variance of optimal inflation in the Justiniano, Primiceri & Tambalotti (2013) model explained by truncating the number of MA lags. Blue: $\rho = 0$. Red: $\rho = 0.61$.

In Figure 4 we plot the proportion of the variance of optimal inflation that is explained by this truncated process for $K = 0, \ldots, 16$, and $\rho \in \{0, 0.61\}$.[5] A multiple shock ARMA(1,1) process already explains over 90% of the variance of optimal inflation, while a multiple shock ARMA(1,2) explains over 95%. Thus, optimal inflation in

[5] $\rho = 0.61$ is the value of ρ that minimises the variance of $\sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k}$. I.e., it is the value of ρ that would be estimated by OLS using an infinite sample of observations from optimal inflation.

plausible models can be well approximated by relatively simple inflation dynamics. More broadly, it seems that thinking about optimal monetary policy in terms of inflation dynamics is a productive approach.

Appendix D Other solutions to the ZLB

D.1 Price level real rate rules

One way to improve the performance of real rate rules near the ZLB is to replace the response to inflation with a response to the price level. Holden (2021) shows that responding to the price level is a robust way to ensure the existence of a unique solution with the ZLB, at least given that inflation does not converge to the deflationary steady state. We discuss how to rule out convergence to the deflationary steady state in Subsection 4.3 of the paper.

Price level rules rule out self-fulfilling temporary jumps to the ZLB as under a price level rule, the deflation during the bound period must be made up for by high inflation after exiting the bound. Thus, expected inflation is high in the last period at the bound, which via the Fisher equation, implies nominal interest rates should be high that period as well, unless real rates are still very low. This unwinds non-fundamental ZLB spells, as in a non-fundamental jump to the bound, real rates are unlikely to move enough to drive the economy to the ZLB on their own.

Incorporating the ideas from the Subsection 4.2, a variable target price level real rate rule takes the form:

$$i_t = \max\{0, r_t + \mathbb{E}_t \breve{p}_{t+1}^* - \breve{p}_t^* + \theta(p_t - \breve{p}_t^*)\},$$

with:

$$\breve{p}_t^* = \breve{p}_{t-1}^* + \max\{-r_{t-1}, (1-\varrho)(p_t^* - \breve{p}_{t-1}^*) + \varrho(p_t^* - p_{t-1}^*)\},$$

where p_t is the logarithm of the price level (so $\pi_t = p_t - p_{t-1}$), p_t^* is the price level target, $\theta > 0$ controls the response to price deviations and $\varrho \in [0,1)$ controls the speed with which $\breve{p}_t^*$ returns to p_t^* following a constrained spell. This has a solution in which $p_t = \breve{p}_t^*$ for all t, since if this holds, then from the monetary rule:

$$i_t - r_t = \max\{-r_t, \mathbb{E}_t\breve{p}_{t+1}^* - \breve{p}_t^*\} = \mathbb{E}_t\breve{p}_{t+1}^* - \breve{p}_t^* = \mathbb{E}_t\pi_{t+1}$$

as $\mathbb{E}_t\breve{p}_{t+1}^* - \breve{p}_t^* \geq -r_t$ by the definition of $\breve{p}_{t+1}^*$, so the Fisher equation holds as required.

Note that $\theta > 0$ is sufficient for determinacy in the absence of the ZLB, since then the monetary rule and Fisher equation imply that:

$$(1+\theta)(p_t - \breve{p}_t^*) = \mathbb{E}_t(p_{t+1} - \breve{p}_{t+1}^*).$$

Thus, price level rules have the same advantage of smoothed rules in not requiring $\phi > 1$. Convincing agents that $\theta > 0$ is likely easier than convincing them that $\phi > 1$, as argued in Subsection 1.5. Furthermore, just like standard (inflation) real rate rules, price level real rate rules are robust, since away from the ZLB, price level determination is completely independent of the real interest rate or the rest of the model. Their chief advantage over standard real rate rules is in avoiding the multiplicity of transition paths highlighted by Holden (2021). In fact, they would manage to do this even had we set $\breve{p}_t^* := p_t^*$.

Additionally, since on the equilibrium path, the Fisher equation under a price level real rate rule implies $i_t - r_t = \mathbb{E}_t\breve{p}_{t+1}^* - p_t$, these rules also guarantee that open market operations can affect $i_t - r_t$ even when expectations are fixed at their values on the equilibrium path. This ensures complete robustness to the concern raised in Subsection 2.4, giving another argument for favouring price level real rate rules.

D.2 Perpetuity real rate rules

An even more robust solution to the problems caused by the ZLB is for the central bank to intervene in a market which does not have an equivalent to the ZLB. Perpetuities (also called "consols") are one such asset. For suppose that nominal interest rates were expected to be at i for all time. Then the price of a perpetuity would be $\frac{1}{i}$.[6] Thus, any finite, positive, perpetuity price is consistent with at least one path for future nominal interest rates. In other words, there is no upper or lower bound on the price of a perpetuity.

Note that the central bank does not strictly need the treasury to issue perpetuities in order to implement a perpetuity real rate rule. Since central banks in developed nations are generally believed to be extremely long-lived institutions, the central bank can issue perpetuities itself. As central banks can always print money to pay the coupon, central banks may be one of the only institutions that could be trusted to pay coupons for ever. Central banks may also decide to trust the perpetuities issued by some selected private banks, even if these will always carry some default risk. If the central bank views default as very unlikely in the short to medium term, then such default risk may not substantially distort pricing.

In the below, we will call standard perpetuities "nominal perpetuities". To implement a real rate rule on perpetuities, we will also need there to be a corresponding "real perpetuity" traded in the economy. In particular, we suppose that one unit of the period t nominal perpetuity bought at t returns \$1 at $t+1$, along with one unit of the period $t+1$ nominal perpetuity. On the other hand, one unit of the period t real perpetuity bought at t returns $\$\frac{P_{t+1}}{\Pi^{*t+1}}$ at $t+1$, along with one of the period $t+1$ real perpetuity, where P_{t+1} is the price level in period $t+1$ and $\Pi^* \geq 1$ is the target for the gross inflation rate. The nominal perpetuity trades at a price of $Q_{I,t}$ at t, whereas the real perpetuity trades at a price of $Q_{R,t}$ at t.

If we write Ξ_{t+1} for the real SDF between periods t and $t+1$, and $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ for gross inflation between these periods, then the price of these two perpetuities must satisfy:

$$Q_{I,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} [Q_{I,t+1} + 1], \qquad Q_{R,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} \left[Q_{R,t+1} + \frac{P_{t+1}}{\Pi^{*t+1}} \right].$$

[6] This is correct under continuous time with a continuous flow of coupons, and approximately correct under discrete time, as we will see below.

The real perpetuity price could be non-stationary due to the potential unit root in the logarithm of the price level, so it is helpful to define a detrended version. In particular, let:

$$\hat{Q}_{R,t} := Q_{R,t}\frac{\Pi^{*t}}{P_t} = \mathbb{E}_t\frac{\Xi_{t+1}}{\Pi^*}\left[\hat{Q}_{R,t+1}+1\right].$$

Rewritten in this way, the analogy between the pricing of nominal and real perpetuities is clear. If $\Pi_t = \Pi^*$ for all t, then $Q_{I,t} = \hat{Q}_{R,t}$ for all t as well. If inflation and the SDF are stationary, then $\hat{Q}_{R,t}$ and $Q_{I,t}$ will admit a stationary solution.

We also assume that one period nominal bonds are traded in the economy, with gross return I_t. As in Subsection 2.1, the pricing for these bonds must satisfy:

$$I_t\mathbb{E}_t\frac{\Xi_{t+1}}{\Pi_{t+1}} = 1.$$

We can now redo the argument of this subsection's initial paragraph, slightly more formally. So, suppose that the gross nominal interest rate I_t is pegged at the constant level I (which may be inconsistent with the inflation target of Π^*). Then, the pricing equation for nominal perpetuities has a solution in which $Q_{I,t} = Q_I$ for all t, with $Q_I = I^{-1}[Q_I + 1]$, since $\mathbb{E}_t\frac{\Xi_{t+1}}{\Pi_{t+1}} = I^{-1}$, for all t. Thus, $Q_I = \frac{1}{I-1}$. As $I \to 1$ (the ZLB), $Q_I \to \infty$, while as $I \to \infty$, $Q_I \to 0$. Thus, in line with our initial argument, any finite, positive, nominal perpetuity price is consistent with at least one possible path for nominal rates, no matter the dynamics of the real SDF. This ensures that the central bank can set the nominal perpetuity price to an arbitrary level, without any constraints. We do not need the real perpetuity price to be unbounded in this manner, as the central bank will not intervene in real perpetuity markets.

The reader might worry that a bound on nominal perpetuity prices could enter another way. Suppose that nominal perpetuity prices were known at least one period in advance (e.g., because there is no uncertainty), and that money is available to trade. Then it would be the case that $Q_{I,t+1} + 1 \geq Q_{I,t}$, else nominal perpetuities would have return strictly dominated by that of cash. This inequality is an immediate consequence of $I_t \geq 1$ though, when $Q_{I,t+1}$ is known at t. $I_t \geq 1$ implies $\frac{Q_{I,t}}{I_t} \leq Q_{I,t}$, so:

$$Q_{I,t}\mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = \frac{Q_{I,t}}{I_t} \leq Q_{I,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} [Q_{I,t+1} + 1],$$

which implies $Q_{I,t+1} + 1 \geq Q_{I,t}$ if $Q_{I,t+1}$ is known at t. Thus, the bound on one period nominal rates is all that really matters, and we have already showed that this bound does not imply a bound on $Q_{I,t}$. Intuitively, $Q_{I,t+1} + 1 \geq Q_{I,t}$ is not a constraint on $Q_{I,t}$ as $Q_{I,t+1}$ is endogenous.

We can now introduce our perpetuity real rate rule. We suppose that the central bank intervenes in nominal perpetuity markets to ensure:

$$Q_{I,t} = \hat{Q}_{R,t} \left(\frac{\Pi_t}{\Pi^*}\right)^{-\psi},$$

for some exponent $\psi \in \mathbb{R}$. While $\psi > 0$ may seem natural (so that high inflation results in low bond prices and thus high interest rates), we do not impose this.

We analyse the resulting dynamics via log-linearizing around the steady-state with inflation at Π^*.[7] In particular, suppose that:

$$Q_{I,t} = Q \exp q_{I,t}, \qquad \hat{Q}_{R,t} = Q \exp q_{R,t},$$
$$\Xi_t = \Xi \exp \xi_t, \qquad \Pi_t = \Pi^* \exp \pi_t,$$

where $Q := \frac{1}{I^*-1}$, with $I^* := \frac{\Pi^*}{\Xi}$. We assume $\Xi < 1$, so $I^* > 1$. Then to a first order approximation around $q_{I,t} = q_{R,t} = \xi_t = \pi_t = 0$:

$$q_{I,t} = \mathbb{E}_t \left[\xi_{t+1} - \pi_{t+1} + \frac{\Xi}{\Pi^*} q_{I,t+1}\right], \qquad q_{R,t} = \mathbb{E}_t \left[\xi_{t+1} + \frac{\Xi}{\Pi^*} q_{R,t+1}\right],$$
$$q_{I,t} = q_{R,t} - \psi\pi_t.$$

Thus:

$$\psi\pi_t = q_{R,t} - q_{I,t} = \mathbb{E}_t \left[\pi_{t+1} + \frac{\Xi}{\Pi^*}(q_{R,t+1} - q_{I,t+1})\right] = \mathbb{E}_t \left[\pi_{t+1} + \frac{\Xi}{\Pi^*}\psi\pi_{t+1}\right].$$

Hence, if we define $\phi := \psi\left[1 + \frac{\Xi}{\Pi^*}\psi\right]^{-1}$, we then have that:

$$\phi\pi_t = \mathbb{E}_t \pi_{t+1},$$

[7] While it would ideally be better to examine these determinacy questions in a fully non-linear model, this is not tractable. We take comfort from the fact that even Cochrane (2011) primarily relies on linearized models.

just as when one period bonds are used. With $\phi > 1$, this has the unique stationary solution $\pi_t = 0$ (so $\Pi_t = \Pi^*$), as usual. The crucial difference is that with the perpetuity real rate rule, this is achieved without violating the ZLB.

As a final observation, note that our definition of ϕ implies that $\psi = -\phi\left[\frac{\Xi}{\Pi^*}\phi - 1\right]^{-1}$, so, for sufficiently large ϕ ($\phi > I^* = \frac{\Pi^*}{\Xi}$) $\psi < -\frac{\Pi^*}{\Xi} < 0$. Thus, under a perpetuity real rate rule with sufficiently large ϕ, the central bank will raise nominal perpetuity prices in response to high inflation. This sign becomes more intuitive once money flows are considered. While if the central bank buys perpetuities, they are raising the money supply in the period of purchase, in every subsequent period they are reducing the money supply, as the private sector must pay coupons back to the central bank. Given the forward-looking nature of inflation determination, it is this long-run reduction which is crucial.

Appendix E Proofs and supplemental results

E.1 Phillips curve based forecasting with ARMA(1,1) policy shocks

As before, we have the monetary rule:

$$i_t = r_t + \phi\pi_t + \zeta_t,$$

which combined with the Fisher equation gives:

$$\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \zeta_t.$$

Suppose ζ_t follows the ARMA(1,1) process:

$$\zeta_t = \rho_\zeta\zeta_{t-1} + \varepsilon_{\zeta,t} + \theta_\zeta\varepsilon_{\zeta,t-1}, \qquad \varepsilon_{\zeta,t} \sim N(0, \sigma_\zeta^2)$$

with $\rho_\zeta, \theta_\zeta \in (-1,1)$. Then from matching coefficients, with $\phi > 1$ we have the unique solution:

$$\pi_t = -\frac{1}{\phi - \rho_\zeta}\left[\zeta_t + \frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right].$$

Thus:

$$\pi_t - \rho_\zeta\pi_{t-1} = -\frac{1}{\phi - \rho_\zeta}\left(1 + \frac{\theta_\zeta}{\phi}\right)\left[\varepsilon_{\zeta,t} + \frac{\phi - \rho_\zeta}{\phi + \theta_\zeta}\theta_\zeta\varepsilon_{\zeta,t-1}\right],$$

so π_t also follows an ARMA(1,1) process. Suppose for now that $-\rho_\zeta \leq \theta_\zeta$, which is likely to be satisfied in reality as we expect ρ_ζ to be large and positive, while θ_ζ should be close to zero. (For example, Dotsey, Fujita & Stark (2018) find that an IMA(1,1) model fits inflation well, in which case $-\rho_\zeta = -1 < \theta_\zeta$ as required.) Then $0 < \frac{\phi-\rho_\zeta}{\phi+\theta_\zeta} < 1$, so $\left|\frac{\phi-\rho_\zeta}{\phi+\theta_\zeta}\theta_\zeta\right| < 1$ meaning the process for inflation is invertible. With inflation following an invertible linear process, the full-information optimal forecast of π_{t+1} is a linear combination of $\pi_t, \pi_{t-1}, \ldots$. In particular, as before x_t is not useful.

In the unlikely case in which $-\rho_\zeta > \theta_\zeta$, of if the forecaster's information set $\mathcal{I}_t$ is smaller than $\{\pi_t, x_t, \pi_{t-1}, x_{t-1}, \ldots\}$,[8] then x_t may contain some useful information. Combining the solution for inflation with the Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t,$$

gives:

$$\begin{aligned} x_t &= -\frac{1}{\kappa}\left[\frac{1-\beta\rho_\zeta}{\phi-\rho_\zeta}\left(\zeta_t + \frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right) - \beta\frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t \\ &= \frac{1}{\kappa}\left[(1-\beta\rho_\zeta)\pi_t + \beta\frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t. \end{aligned}$$

In this case, it is possible that $\mathbb{E}[\pi_{t+1}|\mathcal{I}_t] \neq \mathbb{E}[\pi_{t+1}|\mathcal{I}_{t-1}, \pi_t]$ as x_t provides an independent signal about $\varepsilon_{\zeta,t}$.

There are two important special cases. If $\omega_t = 0$, and the forecaster knows this, then:

$$\varepsilon_{\zeta,t} = \frac{\phi}{\beta\theta_\zeta}\left[\kappa x_t - (1-\beta\rho_\zeta)\pi_t\right],$$

so:

$$\zeta_t = -\left(\phi - \frac{1}{\beta}\right)\pi_t - \frac{\kappa}{\beta}x_t,$$

which enables the forecaster to form the full-information optimal forecast:

$$\mathbb{E}_t \pi_{t+1} = -\frac{1}{\phi-\rho_\zeta}\left(\rho_\zeta \zeta_t + \theta_\zeta \varepsilon_{\zeta,t}\right) = \frac{1}{\beta}(\pi_t - \kappa x_t).$$

(This formula also follows immediately from the Phillips curve.) Note that the output gap has what Dotsey, Fujita & Stark (2018) call the "wrong" sign, meaning Phillips

[8] We nonetheless assume that π_t and x_t are in $\mathcal{I}_t$.

curve based forecasting regressions may have surprising results. However, in the general case in which ω_t has positive variance, then output's signal about $\varepsilon_{\zeta,t}$ will be polluted by the noise from ω_t, making it much less informative. Indeed, with ϕ large, as we expect, then $\frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}$ will have low variance, making it more likely that it is drowned out by the noise from ω_t.

The second important special case is when $\varepsilon_{\zeta,t} = 0$, and again the forecaster knows this. In this case, much as in the main text:

$$\mathbb{E}_t\pi_{t+1} = \rho_\zeta\pi_t - \frac{1}{\phi - \rho_\zeta}\left(1 + \frac{\theta_\zeta}{\phi}\right)\left[\mathbb{E}_t\varepsilon_{\zeta,t+1} + \frac{\phi - \rho_\zeta}{\phi + \theta_\zeta}\theta_\zeta\varepsilon_{\zeta,t}\right] = \rho_\zeta\pi_t,$$

so x_t is unhelpful.

The general case will inherit aspects of these two special cases, as well as the case in which π_t's stochastic process was invertible. Inflation and its lags will certainly help forecast inflation, but the output gap may also provide a little extra information, possibly with the "wrong" sign.

E.2 Robustness to non-unit responses to real interest rates

Suppose that the central bank is unable to respond with a precise unit coefficient to real interest rates, so instead follows the monetary rule:

$$i_t = (1 + \gamma)r_t + \phi\pi_t + \zeta_t,$$

where $\gamma \in \mathbb{R}$ is some small value giving the departure from unit responses.

For simplicity, suppose the rest of the model takes the same form as in Subsection 1.2, with:

$$x_t = \delta\mathbb{E}_t x_{t+1} - \varsigma(r_t - n_t),$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa x_t + \kappa\omega_t,$$

$$i_t = r_t + \mathbb{E}_t\pi_{t+1}.$$

We suppose $\phi > 1$, but do not make any assumptions on the signs of $\delta, \beta, \kappa, \varsigma, \gamma$, beyond assuming that $\varsigma \neq 0$ (so monetary policy has some effect on the output gap) and $\kappa \neq 0$ (so monetary policy has some effect on inflation, via the output gap).

Combining the monetary rule with the Fisher equation gives:

$$\mathbb{E}_t \pi_{t+1} = \gamma r_t + \phi \pi_t + \zeta_t,$$

so:

$$r_t = \frac{1}{\gamma} \left(\mathbb{E}_t \pi_{t+1} - \phi \pi_t - \zeta_t \right),$$

meaning:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \frac{\varsigma}{\gamma} \left(\mathbb{E}_t \pi_{t+1} - \phi \pi_t \right) + \varsigma n_t + \frac{\varsigma}{\gamma} \zeta_t.$$

Then, since:

$$\mathbb{E}_t \pi_{t+1} = \frac{1}{\beta} \pi_t - \frac{\kappa}{\beta} x_t - \frac{\kappa}{\beta} \omega_t,$$

we have that:

$$\mathbb{E}_t x_{t+1} = \left(\frac{1}{\delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} \right) x_t - \frac{\varsigma}{\delta \gamma} \left(\phi - \frac{1}{\beta} \right) \pi_t - \frac{\varsigma}{\delta \gamma} \left(\gamma n_t + \zeta_t + \frac{\kappa}{\beta} \omega_t \right).$$

Woodford (2003) (Addendum to Chapter 4, Proposition C.1) proves that this model is determinate if and only if both eigenvalues of the matrix:

$$M := \begin{bmatrix} \frac{1}{\delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} & -\frac{\varsigma}{\delta \gamma} \left(\phi - \frac{1}{\beta} \right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

are outside of the unit circle, which in turn is proven to hold if and only if EITHER: Case I: $1 < \det M$, $0 < 1 + \det M - \operatorname{tr} M$, and $0 < 1 + \det M + \operatorname{tr} M$, OR Case II: $0 > 1 + \det M - \operatorname{tr} M$, and $0 > 1 + \det M + \operatorname{tr} M$. Note:

$$\det M = \frac{1}{\beta \delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} \phi,$$
$$\operatorname{tr} M = \frac{1}{\delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} + \frac{1}{\beta}.$$

Thus, Case I requires:

$$1 < \det M = \frac{1}{\beta \delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} \phi,$$
$$0 < 1 + \det M - \operatorname{tr} M = \frac{(1 - \beta)(1 - \delta)}{\beta \delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} (\phi - 1),$$
$$\text{and } 0 < 1 + \det M + \operatorname{tr} M = \frac{(1 + \beta)(1 + \delta)}{\beta \delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} (1 + \phi).$$

And Case II requires:

$$
\begin{aligned}
0 > 1 + \det M - \operatorname{tr} M &= \frac{(1-\beta)(1-\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(\phi - 1), \\
\text{and } 0 > 1 + \det M + \operatorname{tr} M &= \frac{(1+\beta)(1+\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(1 + \phi).
\end{aligned}
$$

To see when these conditions are satisfied, first suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} < 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = -\frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then if γ is sufficiently small in magnitude, it is immediately clear that all three conditions of Case I are satisfied, since $\phi > 0$, $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$
|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \dfrac{\phi - 1}{\max\{0, -(\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \dfrac{1 + \phi}{\max\{0, -(\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{array} \right\}.
$$

Alternatively, suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} > 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = \frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then, similarly, if γ is sufficiently small in magnitude, both conditions of Case II are satisfied, since $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$
|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi - 1}{\max\{0, (\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \dfrac{1 + \phi}{\max\{0, (\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{array} \right\}.
$$

Thus, it is always sufficient for determinacy that:

$$
|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \dfrac{\phi - 1}{|(1-\beta)(1-\delta)|}, \\ \dfrac{1 + \phi}{|(1+\beta)(1+\delta)|} \end{array} \right\}.
$$

Since the right-hand side is strictly positive, there is a positive measure of γ for which we have determinacy.

E.3 Real-time learning of Phillips curve coefficients

We start by assuming that the central bank knows the Phillips curve coefficients. A close examination of this case will lead to a natural learning scheme for when the central bank does not know these coefficients.

As in the main text, suppose the central bank is using the rule:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x \big[x_t - \kappa^{-1} \big[\pi_t - \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t \pi_{t+1} - \tilde{\beta}\varrho_\pi \pi_{t-1} \big] \big] + \zeta_t,$$

and that the model also contains the Phillips curve:

$$\pi_t = \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t \pi_{t+1} + \tilde{\beta}\varrho_\pi \pi_{t-1} + \kappa x_t + \kappa \omega_t,$$

and the Fisher equation:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}.$$

We suppose that ζ_t follows the ARMA(1,1) process:

$$\zeta_t = \rho_\zeta \zeta_{t-1} + \varepsilon_{\zeta,t} + \theta_\zeta \varepsilon_{\zeta,t-1}, \qquad \varepsilon_{\zeta,t} \sim N(0, \sigma_\zeta^2),$$

with $\rho_\zeta, \theta_\zeta \in (-1,1)$, and for simplicity, we suppose that $\omega_t = \varepsilon_{\omega,t}$, where $\varepsilon_{\omega,t} \sim N(0, \sigma_\omega^2)$.

From combining all the above equations, we have that if $\phi_\pi > 1$, there is a unique solution with:

$$\pi_t = -\frac{1}{\phi_\pi - \rho_\zeta}\left[\zeta_t + \frac{\theta_\zeta}{\phi_\pi}\varepsilon_{\zeta,t} \right] + \frac{\phi_x}{\phi_\pi}\varepsilon_{\omega,t}.$$

Thus, if we define:

$$m_0 := \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho_\zeta)}\left[\tilde{\beta}(1-\varrho_\pi)(\rho_\zeta + \theta_\zeta) - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right) \right],$$

$$m_1 := \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho_\zeta)}\left[[\tilde{\beta}(1-\varrho_\pi)\rho_\zeta - 1](\rho_\zeta + \theta_\zeta) + \tilde{\beta}\varrho_\pi \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right) \right],$$

$$m_2 := \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho_\zeta)}\big[[\tilde{\beta}(1-\varrho_\pi)\rho_\zeta - 1]\rho_\zeta + \tilde{\beta}\varrho_\pi \big](\rho_\zeta + \theta_\zeta),$$

then by the Phillips curve $m_0 = \mathbb{E} x_t \varepsilon_{\zeta,t}$, $m_1 = \mathbb{E} x_t \varepsilon_{\zeta,t-1}$ and $m_2 = \mathbb{E} x_t \varepsilon_{\zeta,t-2}$. Also note that:

$$
\begin{aligned}
\kappa &= \frac{\sigma_\zeta^2}{\phi_\pi-\rho_\zeta}\frac{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}{\rho_\zeta\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)m_0-\left((\rho_\zeta+\theta_\zeta)m_1-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)}, \\
\tilde{\beta} &= \frac{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)\left(m_0-(\rho_\zeta m_1-m_2)\right)-\frac{\phi_\pi+\theta_\zeta}{(\rho_\zeta+\theta_\zeta)\phi_\pi}\left((\rho_\zeta+\theta_\zeta)m_1-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)}{\rho_\zeta\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)m_0-\left((\rho_\zeta+\theta_\zeta)m_1-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)}, \\
\varrho_\pi &= -\frac{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)(\rho_\zeta m_1-m_2)}{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)\left(m_0-(\rho_\zeta m_1-m_2)\right)-\frac{\phi_\pi+\theta_\zeta}{(\rho_\zeta+\theta_\zeta)\phi_\pi}\left((\rho_\zeta+\theta_\zeta)m_1-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)}.
\end{aligned}
$$

In other words, once the central bank knows m_0, m_1 and m_2 they can infer the parameters of the Phillips curve from the known properties of their monetary rule and monetary shock. This is essentially an instrumental variables regression. We are using $\varepsilon_{\zeta,t}$, $\varepsilon_{\zeta,t-1}$ and $\varepsilon_{\zeta,t-2}$ as instruments for $\mathbb{E}_t\pi_{t+1}$, π_t and π_{t-1} in a regression of the output gap on those variables. This works as long as $\theta_\zeta \neq 0$, else $\mathbb{E}_t\pi_{t+1}$ and π_t are colinear.

If the central bank does not know the true values of κ, $\tilde{\beta}$ and ϱ_π, we suppose they dynamically update estimates of m_0, m_1 and m_2 using the following decreasing gain learning rules (for $t > 0$):

$$
\begin{aligned}
m_{0,t} &= m_{0,t-1} + t^{-1}(x_t\varepsilon_{\zeta,t} - m_{0,t-1}), \\
m_{1,t} &= m_{1,t-1} + t^{-1}(x_t\varepsilon_{\zeta,t-1} - m_{1,t-1}), \\
m_{2,t} &= m_{2,t-1} + t^{-1}(x_t\varepsilon_{\zeta,t-2} - m_{2,t-1}),
\end{aligned}
$$

where $\iota \in (0,1]$ is a gain parameter. Then they can use the monetary rule:

$$
i_t = r_t + \phi_\pi\pi_t + \phi_x[x_t + q_{1,t-1}\mathbb{E}_t\pi_{t+1} + q_{0,t-1}\pi_t + q_{-1,t-1}\pi_{t-1}] + \zeta_t,
$$

where:

$$
\begin{aligned}
q_{1,t} &:= \frac{\phi_\pi-\rho_\zeta}{\sigma_\zeta^2}\frac{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)m_{0,t}-\frac{\phi_\pi+\theta_\zeta}{(\rho_\zeta+\theta_\zeta)\phi_\pi}\left((\rho_\zeta+\theta_\zeta)m_{1,t}-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)m_{2,t}\right)}{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}, \\
q_{0,t} &:= -\frac{\phi_\pi-\rho_\zeta}{\sigma_\zeta^2}\frac{\rho_\zeta\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)m_{0,t}-\left((\rho_\zeta+\theta_\zeta)m_{1,t}-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)m_{2,t}\right)}{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}, \\
q_{-1,t} &:= -\frac{\phi_\pi-\rho_\zeta}{\sigma_\zeta^2}\frac{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)(\rho_\zeta m_{1,t}-m_{2,t})}{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}.
\end{aligned}
$$

This is reasonable, as if $m_{0,t-1} \approx m_0$, $m_{1,t-1} \approx m_1$ and $m_{2,t-1} \approx m_2$ then $q_{1,t-1} \approx \kappa^{-1}\tilde{\beta}(1-\varrho_\pi)$, $q_{0,t-1} \approx -\kappa^{-1}$ and $q_{-1,t-1} \approx \kappa^{-1}\tilde{\beta}\varrho_\pi$, so this monetary rule is approximately the same as the full information one previously considered. Using lagged estimates ($q_{1,t-1}$ not $q_{1,t}$ etc.) in the monetary rule reflects central bank information (processing) delays and simplifies the model's solution. It is also a common assumption in the reduced form learning literature (Evans & Honkapohja 2001).

With the new monetary rule, the model is no-longer linear. As a result, the exact solution is analytically intractable. However, we are only really interested in asymptotic dynamics. If $m_{0,t} \to m_0$, $m_{1,t} \to m_1$ and $m_{2,t} \to m_2$ as $t \to \infty$ then we know the asymptotic solution will be the stable full information one we found previously. We will analyse the system's behaviour with help from the stochastic approximation tools frequently used in the reduced form learning literature (Evans & Honkapohja 2001). These tools only require a zeroth order approximation in t^{-1} to the dynamics of x_t and π_t.[9] Intuitively, this is because x_t (hence π_t) enters the law of motion for $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ multiplied by t^{-1}, so a zeroth order approximation to the dynamics of x_t and π_t in t^{-1} delivers a first order approximation to the dynamics of $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ in t^{-1}.

We conjecture a time-varying coefficients solution with:

$$\pi_t = A_{t-1}\zeta_t + B_{t-1}\varepsilon_{\zeta,t} + C_{t-1}\varepsilon_{\omega,t} + D_{t-1}\pi_{t-1} + O(t^{-1}),$$

where we conjecture $A_t = A_{t-1} + O(t^{-1})$, $B_t = B_{t-1} + O(t^{-1})$, $C_t = C_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$. Substituting this into the monetary rule, Fisher equation and Phillips curve implies:

[9] Given certain regularity conditions on the higher order terms. These conditions will be satisfied here, at least providing we restrict $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ to a small enough open set around m_0, m_1 and m_2, using a so called projection facility.

$$\begin{aligned}
&[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t-1}]A_t(\rho_\zeta\zeta_t+\theta_\zeta\varepsilon_{\zeta,t}) \\
&\qquad = [\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t-1}-[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t-1}]D_t][A_{t-1}\zeta_t \\
&\qquad + B_{t-1}\varepsilon_{\zeta,t}+C_{t-1}\varepsilon_{\omega,t}+D_{t-1}\pi_{t-1}]+\phi_x[q_{-1,t-1}-\kappa^{-1}\tilde{\beta}\varrho_\pi]\pi_{t-1}-\phi_x\varepsilon_{\omega,t} \\
&\qquad + \zeta_t+O(t^{-1}).
\end{aligned}$$

Matching terms and using $A_t = A_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$ then gives that:

$$\begin{aligned}
&[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]A_t\rho_\zeta \\
&\qquad = [\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t}-[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]D_t]A_t+1 \\
&\qquad + O(t^{-1}), \\
&[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]A_t\theta_\zeta \\
&\qquad = [\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t}-[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]D_t]B_t+O(t^{-1}), \\
&0 = [\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t}-[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t-1}]D_t]C_t-\phi_x+O(t^{-1}), \\
&0 = [\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t}-[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]D_t]D_t+\phi_x[q_{-1,t}-\kappa^{-1}\tilde{\beta}\varrho_\pi] \\
&\qquad + O(t^{-1}).
\end{aligned}$$

The final equation has two roots, but we know we need to pick the one that gives $D_t \to 0$ as $\phi_x \to 0$. Now if $q_{0,t}$ is sufficiently close to q_0, then $\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} > 0$, so:

$$D_t = \frac{(\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t}) - \sqrt{\begin{matrix}(\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t})^2 \cdots \\ +4\phi_x[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}][q_{-1,t}-\kappa^{-1}\tilde{\beta}\varrho_\pi]\end{matrix}}}{2[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]}$$
$$+ O(t^{-1}),$$

and:

$$\begin{aligned}
A_t &= \left[[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}](D_t+\rho_\zeta)-(\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t})\right]^{-1}+O(t^{-1}), \\
B_t &= \frac{\theta_\zeta[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]A_t}{\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t}-[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]D_t}+O(t^{-1}), \\
C_t &= \frac{\phi_x}{\phi_\pi+\phi_x\kappa^{-1}+\phi_x q_{0,t}-[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x q_{1,t}]D_t}+O(t^{-1}).
\end{aligned}$$

Since $q_{1,t} = q_{1,t-1} + O(t^{-1})$, $q_{0,t} = q_{0,t-1} + O(t^{-1})$ and $q_{-1,t} = q_{-1,t-1} + O(t^{-1})$, as required we have that $A_t = A_{t-1} + O(t^{-1})$, $B_t = B_{t-1} + O(t^{-1})$, $C_t = C_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$.

Using this result again, we then have that:

$$\begin{aligned} x_t = \kappa^{-1}\Big[&\big[1-\tilde{\beta}(1-\varrho_\pi)\big(D_{t-1}+\rho_\zeta\big)\big]A_{t-1}\zeta_t \\ &+\big[B_{t-1}-\tilde{\beta}(1-\varrho_\pi)\big(A_{t-1}\theta_\zeta+B_{t-1}D_{t-1}\big)\big]\varepsilon_{\zeta,t} \\ &+\big[\big[1-\tilde{\beta}(1-\varrho_\pi)D_{t-1}\big]C_{t-1}-\kappa\big]\varepsilon_{\omega,t} \\ &+\big[\big[1-\tilde{\beta}(1-\varrho_\pi)D_{t-1}\big]D_{t-1}-\tilde{\beta}\varrho_\pi\big]\pi_{t-1}\Big]+O(t^{-1}). \end{aligned}$$

Plugging this into the law of motion for $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ gives a purely backward looking non-linear system in the endogenous states $m_{0,t}$, $m_{1,t}$, $m_{2,t}$ and π_t. This system is of the correct form to be analysed by the stochastic approximation results given in Evans & Honkapohja (2001).

To apply these results, first suppose that for all t, $m_{0,t}=\widehat{m}_0$, $m_{1,t}=\widehat{m}_1$ and $m_{2,t}=\widehat{m}_2$, for some values $\widehat{m}_0$, $\widehat{m}_1$ and $\widehat{m}_2$. Then $q_{1,t}=\hat{q}_1$, $q_{0,t}=\hat{q}_0$ and $q_{-1,t}=\hat{q}_{-1}$ for all t, where:

$$\begin{aligned} \hat{q}_1 &:= \frac{\phi_\pi-\rho_\zeta}{\sigma_\zeta^2}\frac{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)\widehat{m}_0-\frac{\phi_\pi+\theta_\zeta}{(\rho_\zeta+\theta_\zeta)\phi_\pi}\left((\rho_\zeta+\theta_\zeta)\widehat{m}_1-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\widehat{m}_2\right)}{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}, \\ \hat{q}_0 &:= -\frac{\phi_\pi-\rho_\zeta}{\sigma_\zeta^2}\frac{\rho_\zeta\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)\widehat{m}_0-\left((\rho_\zeta+\theta_\zeta)\widehat{m}_1-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\widehat{m}_2\right)}{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}, \\ \hat{q}_{-1} &:= -\frac{\phi_\pi-\rho_\zeta}{\sigma_\zeta^2}\frac{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)(\rho_\zeta\widehat{m}_1-\widehat{m}_2)}{\left(\rho_\zeta+\theta_\zeta-\left(1+\frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}. \end{aligned}$$

Thus, for all t, $A_t=\hat{A}$, $B_t=\hat{B}$, $C_t=\hat{C}$ and $D_t=\widehat{D}$, where:

$$\widehat{D}=\frac{(\phi_\pi+\phi_x\kappa^{-1}+\phi_x\hat{q}_0)-\sqrt{\begin{matrix}(\phi_\pi+\phi_x\kappa^{-1}+\phi_x\hat{q}_0)^2\cdots\\+4\phi_x\big[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x\hat{q}_1\big]\big[\hat{q}_{-1}-\kappa^{-1}\tilde{\beta}\varrho_\pi\big]\end{matrix}}}{2\big[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x\hat{q}_1\big]},$$

and:

$$\begin{aligned} \hat{A} &= \Big[\big[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x\hat{q}_1\big]\big(\widehat{D}+\rho_\zeta\big)-\big(\phi_\pi+\phi_x\kappa^{-1}+\phi_x\hat{q}_0\big)\Big]^{-1}, \\ \hat{B} &= \frac{\theta_\zeta\big[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x\hat{q}_1\big]\hat{A}}{\phi_\pi+\phi_x\kappa^{-1}+\phi_x\hat{q}_0-\big[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x\hat{q}_1\big]\widehat{D}}, \\ \hat{C} &= \frac{\phi_x}{\phi_\pi+\phi_x\kappa^{-1}+\phi_x\hat{q}_0-\big[1+\phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi)-\phi_x\hat{q}_1\big]\widehat{D}}. \end{aligned}$$

So:

$$\pi_t=\hat{A}\zeta_t+\hat{B}\varepsilon_{\zeta,t}+\hat{C}\varepsilon_{\omega,t}+\widehat{D}\pi_{t-1},$$

and:

$$
\begin{aligned}
x_t &= \kappa^{-1}\Big[\big[1-\tilde{\beta}(1-\varrho_\pi)(\widehat{D}+\rho_\zeta)\big]\hat{A}\zeta_t + \big[\hat{B}-\tilde{\beta}(1-\varrho_\pi)(\hat{A}\theta_\zeta+\hat{B}\widehat{D})\big]\varepsilon_{\zeta,t} \\
&\quad + \Big[\big[1-\tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\hat{C}-\kappa\Big]\varepsilon_{\omega,t} + \Big[\big[1-\tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D}-\tilde{\beta}\varrho_\pi\Big]\pi_{t-1}\Big] \\
&= \kappa^{-1}\Big[\big[1-\tilde{\beta}(1-\varrho_\pi)(\widehat{D}+\rho_\zeta)\big]\hat{A}\big[\rho_\zeta\big[\rho_\zeta\zeta_{t-2}+\varepsilon_{\zeta,t-1}+\theta_\zeta\varepsilon_{\zeta,t-2}\big]+\varepsilon_{\zeta,t}+\theta_\zeta\varepsilon_{\zeta,t-1}\big] \\
&\quad + \big[\hat{B}-\tilde{\beta}(1-\varrho_\pi)(\hat{A}\theta_\zeta+\hat{B}\widehat{D})\big]\varepsilon_{\zeta,t} + \Big[\big[1-\tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\hat{C}-\kappa\Big]\varepsilon_{\omega,t} \\
&\quad + \Big[\big[1-\tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D}-\tilde{\beta}\varrho_\pi\Big]\Big[\hat{A}\big[\rho_\zeta\zeta_{t-2}+\varepsilon_{\zeta,t-1}+\theta_\zeta\varepsilon_{\zeta,t-2}\big]+\hat{B}\varepsilon_{\zeta,t-1} \\
&\quad + \hat{C}\varepsilon_{\omega,t-1}+\widehat{D}\big[\hat{A}\zeta_{t-2}+\hat{B}\varepsilon_{\zeta,t-2}+\hat{C}\varepsilon_{\omega,t-2}+\widehat{D}\pi_{t-3}\big]\Big]\Big].
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\mathbb{E}x_t\varepsilon_{\zeta,t} &= \sigma_\zeta^2\kappa^{-1}\Big[\big[1-\tilde{\beta}(1-\varrho_\pi)(\widehat{D}+\rho_\zeta+\theta_\zeta)\big]\hat{A}+\big[1-\tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\hat{B}\Big], \\
\mathbb{E}x_t\varepsilon_{\zeta,t-1} &= \sigma_\zeta^2\kappa^{-1}\Big[\big[1-\tilde{\beta}(1-\varrho_\pi)(\widehat{D}+\rho_\zeta)\big]\hat{A}(\rho_\zeta+\theta_\zeta) \\
&\quad + \Big[\big[1-\tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D}-\tilde{\beta}\varrho_\pi\Big](\hat{A}+\hat{B})\Big], \\
\mathbb{E}x_t\varepsilon_{\zeta,t-2} &= \sigma_\zeta^2\kappa^{-1}\Big[\big[1-\tilde{\beta}(1-\varrho_\pi)(\widehat{D}+\rho_\zeta)\big]\hat{A}\rho_\zeta(\rho_\zeta+\theta_\zeta) \\
&\quad + \Big[\big[1-\tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D}-\tilde{\beta}\varrho_\pi\Big]\big[\hat{A}(\rho_\zeta+\theta_\zeta)+\widehat{D}(\hat{A}+\hat{B})\big]\Big].
\end{aligned}
$$

Now denote by $\mathcal{T}$ the map taking the vector:

$$
\widehat{m}:=\begin{bmatrix}\widehat{m}_0\\ \widehat{m}_1\\ \widehat{m}_2\end{bmatrix}
$$

to the vector:

$$
\mathcal{T}(\widehat{m}):=\begin{bmatrix}\mathbb{E}x_t\varepsilon_{\zeta,t}\\ \mathbb{E}x_t\varepsilon_{\zeta,t-1}\\ \mathbb{E}x_t\varepsilon_{\zeta,t-2}\end{bmatrix}.
$$

Stochastic approximation theory relates the stability of our nonlinear difference equation to the stability of the ODE:

$$
\frac{d\widehat{m}(\tau)}{d\tau}=\mathcal{T}\big(\widehat{m}(\tau)\big)-\widehat{m}(\tau).
$$

The $\mathcal{T}$ map here plays the role usually played by the mapping from the perceived law of motion to the actual law of motion in the reduced form learning literature (Evans & Honkapohja 2001).

We conjecture that:

$$
m:=\begin{bmatrix}m_0\\ m_1\\ m_2\end{bmatrix}
$$

is a locally asymptotically stable point of this ODE. To check this, note that tedious algebra gives that:

$$\left.\frac{\partial \mathcal{T}(\widehat{m})}{\partial \widehat{m}}\right|_{\widehat{m}=m} = \frac{\phi_x}{\kappa\phi_\pi}\begin{bmatrix} 1 & \phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi) & \dfrac{\phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi)}{\phi_\pi - \rho_\zeta} \\ -\tilde{\beta}\varrho_\pi & 1 - \phi_\pi^{-1}\tilde{\beta}\varrho_\pi & \dfrac{\phi_\pi\left[\phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi)\right] - \phi_\pi^{-1}\tilde{\beta}\varrho_\pi}{\phi_\pi - \rho_\zeta} \\ 0 & -\tilde{\beta}\varrho_\pi & \dfrac{\phi_\pi\left[1 - \tilde{\beta}(1-\varrho_\pi)\rho_\zeta\right] - \tilde{\beta}\varrho_\pi}{\phi_\pi - \rho_\zeta} \end{bmatrix}.$$

For simplicity, we assume $\phi_x \geq 0, \phi_\pi \geq 0, \kappa \geq 0, \tilde{\beta} \geq 0, \varrho_\pi \in [0,1), \rho_\zeta \in [0,1)$ and $\phi_\pi \geq \left[\tilde{\beta}(1-\varrho_\pi)\right]^{-1}$. Under these assumptions, the off-diagonal elements of this matrix are all non-positive. Other cases may also go through, but for the sake of brevity we concentrate on this most relevant case. Given these assumptions, applying the Gershgorin circle theorem to the columns of this matrix gives the following upper bound on the real part of the eigenvalues of $\left.\frac{\partial \mathcal{T}(\widehat{m})}{\partial \widehat{m}}\right|_{\widehat{m}=m}$:

$$\frac{\phi_x}{\kappa\phi_\pi}\max\left\{\begin{matrix} 1+\tilde{\beta}\varrho_\pi, \phi_\pi^{-1}\left[\tilde{\beta}(\phi_\pi - \varrho_\pi) + \phi_\pi - 1\right], \\ \dfrac{(1-\phi_\pi^{-1})(\phi_\pi - \tilde{\beta}\varrho_\pi) + \tilde{\beta}(1-\varrho_\pi)\left[1 + \phi_\pi(1-\rho_\zeta)\right] - \phi_\pi^{-1}}{\phi_\pi - \rho_\zeta} \end{matrix}\right\}.$$

The first and second arguments in curly brackets here are both less than $1+\tilde{\beta}$. Taking the derivative of the third argument in curly brackets with respect to ρ_ζ produces an expression whose sign is not a function of ρ_ζ. Thus, the third argument in curly brackets is maximized at either $\rho_\zeta = 0$ or $\rho_\zeta = 1$. In the former case, the argument is less or equal to $1+\tilde{\beta}$ providing $\tilde{\beta} \leq 1$. In the latter case, the argument is less or equal to $1+\tilde{\beta}$ providing that $2(1-\varrho_\pi) \leq \phi_\pi$. Therefore, if $\phi_x \geq 0$, $\phi_\pi \geq 0$, $\kappa \geq 0$, $\tilde{\beta} \in [0,1]$, $\varrho_\pi \in [0,1)$, $\rho_\zeta \in [0,1)$ and:

$$\phi_\pi > \max\left\{\frac{1}{\tilde{\beta}(1-\varrho_\pi)}, 2(1-\varrho_\pi), \frac{\phi_x(1+\tilde{\beta})}{\kappa}\right\},$$

then all of the eigenvalues of $\left.\frac{\partial \mathcal{T}(\widehat{m})}{\partial \widehat{m}}\right|_{\widehat{m}=m}$ are less than one. Consequently, in this case the ODE is locally asymptotically stable, so the stochastic approximation results of Evans & Honkapohja (2001) apply. In particular, if we suppose that $\widehat{m}_0$, $\widehat{m}_1$ and $\widehat{m}_2$ are constrained to remain within a sufficiently small ball around m_0, m_1 and m_2, then the central bank's estimates of the Phillips curve parameters will converge to their true values, and the model's dynamics will converge to the determinate ones under rational expectations.

E.4 Responding to other endogenous variables in a general model

Now, suppose the central bank uses the rule:

$$i_t = r_t + \phi_\pi \pi_t + \iota \phi_z^\top z_t + \phi_\nu^\top \nu_t.$$

Here, z_t is a vector of other endogenous variables, with $z_{t,1} = r_t$, $\iota > 0$ is a scalar governing the strength of response to all of them, and ν_t is an arbitrary exogenous stochastic process (potentially vector valued). As usual, we assume $\phi_\pi > 1$. We also assume without loss of generality that the elements of z_t are all zero in steady state.

Without loss of generality, we suppose that the other endogenous variables satisfy the general linear expectational difference equation:

$$0 = A\mathbb{E}_t z_{t+1} + Bz_t + Cz_{t-1} + d\pi_t + E\nu_t,$$

where the coefficient matrices are such that there is a unique matrix F with eigenvalues in the unit circle such that $F = -(AF + B)^{-1}C$.[10] This condition on F just states that there is no real indeterminacy in the model. Once inflation is determined, so too is z_t. Having the same shock process entering both the monetary rule and the model's other equations is without loss of generality as it is multiplied by $\phi_\nu^\top$ and E respectively.

Now define:

$$G := -A(AF + B)^{-1}.$$

Let L be the lag operator, then note that:

$$(I - GL^{-1})(AF + B)(I - FL) = AL^{-1} + B + CL.$$

Thus, by the model's real determinacy, all of G's eigenvalues must also be inside the unit circle.

---

[10] The lack of terms in $\mathbb{E}_t \pi_{t+1}$ and π_{t-1} is without loss of generality, as such responses can be included by adding an auxiliary variable $z_{t,j}$ with an equation of the form $z_{t,j} = \pi_t$.

In terms of the lag operator, the model to be solved is then:

$$\mathbb{E}_t(1-\phi_\pi^{-1}L^{-1})\pi_t = -\iota\phi_\pi^{-1}\phi_z^\top z_t - \phi_\pi^{-1}\phi_\nu^\top \nu_t,$$

$$\mathbb{E}_t(I-GL^{-1})(AF+B)(I-FL)z_t = -d\pi_t - E\nu_t.$$

Note for future reference that since ϕ_π^{-1}, G and F all have all their eigenvalues in the unit circle, $(1-\phi_\pi^{-1}L^{-1})$, $(I-GL^{-1})$ and $(I-FL)$ are all invertible.

We conjecture a series solution of the form:

$$\pi_t = \sum_{k=0}^{\infty} \iota^k \pi_t^{(k)}, \qquad z_t = \sum_{k=0}^{\infty} \iota^k z_t^{(k)}.$$

Matching terms gives that $\pi_t^{(0)}$ solves:

$$\mathbb{E}_t(1-\phi_\pi^{-1}L^{-1})\pi_t^{(0)} = -\phi_\pi^{-1}\phi_\nu^\top \nu_t,$$

implying that $\pi_t^{(0)}$ is determinate with:

$$\pi_t^{(0)} = -\mathbb{E}_t(1-\phi_\pi^{-1}L^{-1})^{-1}\phi_\pi^{-1}\phi_\nu^\top \nu_t.$$

Similarly, from matching terms in the law of motion for z_t, we have that:

$$\mathbb{E}_t(I-GL^{-1})(AF+B)(I-FL)z_t^{(0)} = -d\pi_t^{(0)} - E\nu_t$$

so $z_t^{(0)}$ is also determinate (by our assumption on A, B and C) with:

$$z_t^{(0)} = -(I-FL)^{-1}(AF+B)^{-1}\mathbb{E}_t(I-GL^{-1})^{-1}\big(d\pi_t^{(0)} - E\nu_t\big).$$

Note that $\pi_t^{(0)}$ can be treated as exogenous for solving for $z_t^{(0)}$, as the causation only runs one way, from $\pi_t^{(0)}$ to $z_t^{(0)}$.

Now suppose that we have established that $\pi_t^{(k)}$ and $z_t^{(k)}$ are determinate for some $k \in \mathbb{N}$, with a determined solution not a function of higher order terms. (We have already proven the base case of $k=0$.) We seek to prove that $\pi_t^{(k+1)}$ and $z_t^{(k+1)}$ are also determinate. Matching terms again gives that:

$$\mathbb{E}_t(1-\phi_\pi^{-1}L^{-1})\pi_t^{(k+1)} = -\phi_\pi^{-1}\phi_z^\top z_t^{(k)},$$

so $\pi_t^{(k+1)}$ is also determinate, with:

$$\pi_t^{(k+1)} = -\mathbb{E}_t(1-\phi_\pi^{-1}L^{-1})^{-1}\phi_\pi^{-1}\phi_z^\top z_t^{(k)},$$

where we used the inductive hypothesis that $z_t^{(k)}$ is already determined, and so it is effectively exogenous for the purpose of determining $\pi_t^{(k+1)}$. Then from matching terms in the law of motion for z_t:

$$\mathbb{E}_t(I-GL^{-1})(AF+B)(I-FL)z_t^{(k+1)} = -d\pi_t^{(k+1)},$$

so $z_t^{(k+1)}$ is also determinate, with:

$$z_t^{(k+1)} = -(I-FL)^{-1}(AF+B)^{-1}\mathbb{E}_t(I-GL^{-1})^{-1}d\pi_t^{(k+1)},$$

much as before. This completes our proof by induction, establishing that there is a series solution of the given form.

The only remaining thing to check is that the series does indeed converge for sufficiently small ι. This follows immediately from the product structure of the solution above, which means that the variances of $z_t^{(k)}$ and $\pi_t^{(k)}$ must be $O(h^k)$ for some $h \geq 1$. Hence for sufficiently small ι, the model is determinate. I.e., given the Taylor principle is satisfied, a sufficiently small response to other endogenous variables will not break determinacy.

E.5 Real rate rules with exogenous targets

We want to prove that even with an exogenous π_t^*, rules in the form of (7) can still mimic the outcomes of any other monetary policy regime.

Suppose that the central bank were to set interest rates in a different (though time invariant) way, for example by using another rule, or by adopting optimal policy under either commitment or discretion, given some objective. For simplicity, suppose further that the economy's equilibrium conditions are linear, e.g., because we are working under a first order approximation. Let $(\varepsilon_{1,t}, \dots, \varepsilon_{N,t})_{t\in\mathbb{Z}}$ be the set of structural shocks in the economy,[11] all of which are assumed mean zero and independent both of each other, and over time. Finally, assume that the central bank's behaviour produces stationary inflation, $\tilde{\pi}_t$, with the $\sim$ denoting that this is inflation under the alternative

[11] This may include sunspot shocks if they are added following Farmer, Khramov & Nicolò (2015).

monetary regime. Then, by linearity and stationarity, there must exist a constant $\tilde{\pi}^*$ and coefficients $(\theta_{1,k},\dots,\theta_{N,k})_{k\in\mathbb{N}}$ such that:

$$\tilde{\pi}_t = \tilde{\pi}^* + \sum_{k=0}^{\infty}\sum_{n=1}^{N}\theta_{n,k}\varepsilon_{n,t-k},$$

with $\sum_{k=0}^{\infty}\theta_{n,k}^2 < \infty$ for $n = 1,\dots,N$. So, if the central bank sets:

$$\pi_t^* = \tilde{\pi}^* + \sum_{k=0}^{\infty}\sum_{n=1}^{N}\theta_{n,k}\varepsilon_{n,t-k},$$

(exogenous!) and uses the rule (7), then for all t and in all states of the world, $\pi_t = \pi_t^* = \tilde{\pi}_t$. Moreover, this implies in turn that all the endogenous variables in the two economies must be identical in all periods and in all states of the world.

To see this final claim, let z_t and $\tilde{z}_t$ be vectors stacking the endogenous variables other than inflation in the economy with our rule and the economy with the alternative rule, respectively, with $z_{t,1} = r_t$ and $\tilde{z}_{t,1} = \tilde{r}_t$. We assume without loss of generality that the elements of z_t and $\tilde{z}_t$ are all zero in steady state.

By linearity, without loss of generality, the equations other than the monetary rule or monetary policy first order condition must have the form:[12]

$$0 = A\mathbb{E}_t z_{t+1} + Bz_t + Cz_{t-1} + d\pi_t + \sum_{n=1}^{N} f_n\varepsilon_{n,t}\,, \tag{12}$$

in the economy with our rule, and they must have the form:

$$0 = A\mathbb{E}_t\tilde{z}_{t+1} + B\tilde{z}_t + C\tilde{z}_{t-1} + d\tilde{\pi}_t + \sum_{n=1}^{N} f_n\varepsilon_{n,t},$$

in the economy with the alternative rule. (Here, A, B and C are square matrices, while d and $f_1,\dots,f_N$ are vectors.) Since $\pi_t = \tilde{\pi}_t$ for all t, $z_t = \tilde{z}_t$ must solve equation (12) for all t. It will be the unique solution providing the model has no source of indeterminacy other than perhaps monetary policy. For example, in a three equation NK model, given that $\pi_t \equiv \tilde{\pi}_t$, the Phillips curve implies that the output gap must agree in the two economies, thus the Euler equation then implies that the interest rate must also agree.

[12] The lack of terms in $\mathbb{E}_t\pi_{t+1}$ and π_{t-1} is without loss of generality, as such responses can be included by adding an auxiliary variable $z_{t,j}$ with an equation of the form $z_{t,j} = \pi_t$.

To see the uniqueness more formally, suppose that there is a unique matrix F with eigenvalues in the unit circle such that $F = -(AF + B)^{-1}C$. This condition on F just states that there is no real indeterminacy in the model.

Now define:

$$G := -A(AF + B)^{-1}.$$

Let L be the lag operator, then note (as in the previous appendix subsection) that:

$$(I - GL^{-1})(AF + B)(I - FL) = AL^{-1} + B + CL.$$

Thus, by the model's real determinacy, all of G's eigenvalues must also be inside the unit circle. Hence, since G and F all have all their eigenvalues in the unit circle, $(I - GL^{-1})$ and $(I - FL)$ are both invertible.

In terms of the lag operator, the equations determining z_t and $\tilde{z}_t$ are:

$$\begin{aligned}\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t &= -d\pi_t - \sum_{n=1}^{N} f_n \varepsilon_{n,t} \\ &= -d\tilde{\pi}_t - \sum_{n=1}^{N} f_n \varepsilon_{n,t} \\ &= \mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)\tilde{z}_t,\end{aligned}$$

as $\pi_t = \tilde{\pi}_t$ for all t. Consequently:

$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)(z_t - \tilde{z}_t) = 0.$$

Therefore, by the invertibility of $(I - GL^{-1})$, $(AF + B)$ and $(I - FL)$, $z_t = \tilde{z}_t$ for all t, as required. (Expectations drop out as the right-hand side is deterministic.)

The only slight difficulty with setting π_t^* as a function of structural shocks is that the central bank may struggle to observe these shocks. The central bank can certainly observe linear combinations of structural shocks, via estimating a VAR with sufficiently many lags. For variables that are plausibly contemporaneously exogenous, such as commodity prices for a small(ish) economy, this is already sufficient to recover the corresponding structural shock. To infer other shocks, the central bank needs to know more about the structure of the economy. However, we do not need to assume any more than is standard in rational expectations models. Forming rational expectations requires you to know the structure of the economy; if you know this structure, then you know the mapping from the reduced form shocks estimated by a VAR to the model's structural shocks.[13] Additionally, it is common to assume that the central bank responds to an output gap constructed by comparing outcomes to an economy without price rigidity. This already requires the central bank to know the values of all parameters and structural shocks.

E.6 Partially smoothed real rate rules

Suppose that the central bank sets interest rates according to the partially smoothed real rate rule:

$$i_t - r_t = \varrho_i(i_{t-1} - r_{t-1}) + \mathbb{E}_t\pi^*_{t+1} - \varrho_i\mathbb{E}_{t-1}\pi^*_t + (1-\varrho_i)\phi(\pi_t - \pi^*_t),$$

where $\phi > 1$, $\varrho_i < 1$ and where π^*_t is the inflation target. Then, from the Fisher equation:

$$\mathbb{E}_t(\pi_{t+1} - \pi^*_{t+1}) = \varrho_i\mathbb{E}_{t-1}(\pi_t - \pi^*_t) + (1-\varrho_i)\phi(\pi_t - \pi^*_t).$$

Now let $\hat{\pi}_t := \pi_t - \pi^*_t$ and $e_t := \mathbb{E}_t(\pi_{t+1} - \pi^*_{t+1})$. Then we have the system:

$$e_t = \mathbb{E}_t\hat{\pi}_{t+1},$$

$$e_t = \varrho_i e_{t-1} + (1-\varrho_i)\phi\hat{\pi}_t.$$

Equivalently:

$$\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\mathbb{E}_t\begin{bmatrix}\hat{\pi}_{t+1}\\e_t\end{bmatrix} = \begin{bmatrix}0 & 0\\(1-\varrho_i)\phi & \varrho_i\end{bmatrix}\begin{bmatrix}\hat{\pi}_t\\e_{t-1}\end{bmatrix},$$

so, from pre-multiplying by $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$:

$$\mathbb{E}_t\begin{bmatrix}\hat{\pi}_{t+1}\\e_t\end{bmatrix} = \begin{bmatrix}(1-\varrho_i)\phi & \varrho_i\\(1-\varrho_i)\phi & \varrho_i\end{bmatrix}\begin{bmatrix}\hat{\pi}_t\\e_{t-1}\end{bmatrix}.$$

Now:

$$\begin{bmatrix}(1-\varrho_i)\phi & \varrho_i\\(1-\varrho_i)\phi & \varrho_i\end{bmatrix} = \begin{bmatrix}-\frac{\varrho_i}{(1-\varrho_i)\phi} & 1\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & \varrho_i + (1-\varrho_i)\phi\end{bmatrix}\begin{bmatrix}-\frac{\varrho_i}{(1-\varrho_i)\phi} & 1\\1 & 1\end{bmatrix}^{-1}.$$

[13] This mapping may not be unique valued if there are more shocks than observables. However, since we expect a relatively small number of shocks to explain the bulk of business cycle variance, this is unlikely to be problematic in practice.

Thus, if we define:

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} := \begin{bmatrix} -\frac{\varrho_i}{(1-\varrho_i)\phi} & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\pi}_t \\ e_{t-1} \end{bmatrix} = \frac{(1-\varrho_i)\phi}{\varrho_i + (1-\varrho_i)\phi} \begin{bmatrix} -1 & 1 \\ 1 & \frac{\varrho_i}{(1-\varrho_i)\phi} \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ e_{t-1} \end{bmatrix},$$

then:

$$\mathbb{E}_t \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \varrho_i + (1-\varrho_i)\phi \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix}.$$

Now, since $\phi > 1$ and $\varrho_i < 1$, $\varrho_i + (1-\varrho_i)\phi = \phi - \varrho_i(\phi - 1) > 1$. Thus, the unique non-explosive solution for v_t is $v_t = 0$. (Note that v_t must be stationary as $\hat{\pi}_t$ and e_{t-1} must be stationary.) Hence, by the definition of v_t:

$$\hat{\pi}_t = -\frac{\varrho_i}{(1-\varrho_i)\phi} e_{t-1}.$$

So as $e_t = \mathbb{E}_t \hat{\pi}_{t+1}$:

$$e_t = -\frac{\varrho_i}{(1-\varrho_i)\phi} e_t,$$

i.e.:

$$[\varrho_i + (1-\varrho_i)\phi] e_t = 0,$$

so $e_t = 0$, and hence $\hat{\pi}_t = 0$.

Therefore, with $\phi > 1$, $\pi_t = \pi_t^*$ is the unique stationary solution.

Finally, note that the coefficient on $\pi_t - \pi_t^*$ in the original rule was $(1-\varrho_i)\phi$, so for any $\theta > 0$ if we set $\phi := \frac{\theta}{1-\varrho_i}$ then for ϱ_i sufficiently close to 1, $\phi > 1$ as required. Thus, for ϱ_i sufficiently close to 1 a coefficient of $\theta > 0$ on $\pi_t - \pi_t^*$ will do. This links the results of this appendix to those of the main text.

E.7 Dynamics under lag-augmented real rate rules

We are interested in the solution of the expectational difference equation:

$$\mathbb{E}_t \pi_{t+1} = \phi \pi_t + \psi \pi_{t-1}.$$

The two roots of the characteristic equation are given by:

$$\frac{\phi \pm \sqrt{\phi^2 + 4\psi}}{2}.$$

We need to prove that if $\phi > |1-\psi|$, the positive root is strictly greater than 1, while the negative root is in $(-1{,}1)$.

For this, we first need to prove that these roots are real. This follows as:

$$\phi^2 > (1-\psi)^2 = 1 - 2\psi + \psi^2,$$

so:

$$\phi^2 + 4\psi > 1 + 2\psi + \psi^2 = (1+\psi)^2 \geq 0.$$

Next, note that $\phi > |1-\psi| > 1-\psi$, so $\psi > 1-\phi$. Hence:

$$\phi^2 + 4\psi > \phi^2 - 4\phi + 4 = (\phi-2)^2,$$

meaning that:

$$\sqrt{\phi^2 + 4\psi} > |\phi - 2|.$$

Thus:

$$\phi + \sqrt{\phi^2 + 4\psi} > 2,$$

and:

$$\phi - \sqrt{\phi^2 + 4\psi} < 2.$$

Finally, note that $\phi > |1-\psi| > \psi - 1$, so $\psi < \phi + 1$. Hence:

$$\phi^2 + 4\psi < \phi^2 + 4\phi + 4 = (\phi+2)^2,$$

meaning that:

$$\phi + 2 > \sqrt{\phi^2 + 4\psi},$$

so:

$$\phi - \sqrt{\phi^2 + 4\psi} > -2.$$

Hence, we have established that as required:

$$\frac{\phi + \sqrt{\phi^2 + 4\psi}}{2} > 1,$$

while:

$$-1 < \frac{\phi + \sqrt{\phi^2 + 4\psi}}{2} < 1.$$

E.8 Roots of the characteristic equation arising from multiperiod bonds

We are interested in the roots for $\lambda \in \mathbb{C}$ of the characteristic equation:

$$\frac{1}{T}\sum_{k=1}^{T}\lambda^{k+S-L} = \phi,$$

for $T, S, L \in \mathbb{N}$ and $\phi > 1$. We wish to prove that this equation has $\max\{0, -(1+S-L)\}$ roots strictly inside the unit circle and $\max\{0, T+S-L\}$ roots strictly outside of the unit circle. We proceed by cases. (These cases have some overlap, which is inconsequential.)

Case 1: $1 + S - L \geq 0$

Note that in this case, $k + S - L \geq 0$ for all $k \in \{1, \ldots, T\}$. Thus if $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, then by the triangle inequality:

$$\left|\frac{1}{T}\sum_{k=1}^{T}\lambda^{k+S-L}\right| \leq \frac{1}{T}\sum_{k=1}^{T}\left|\lambda^{k+S-L}\right| \leq \frac{1}{T}\sum_{k=1}^{T}1 = 1 < \phi.$$

Hence, in this case, there cannot be any roots weakly inside the unit circle. Thus, by the fundamental theorem of algebra, the equation has $\max\{0, T+S-L\}$ roots all strictly outside the unit circle.

Case 2: $T + S - L \leq 0$

In this case, $-(k+S-L) \geq 0$ for all $k \in \{1, \ldots, T\}$. Suppose $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ and define $\kappa := \lambda^{-1}$, so $|\kappa| \leq 1$. Then again by the triangle inequality:

$$\left|\frac{1}{T}\sum_{k=1}^{T}\lambda^{k+S-L}\right| = \left|\frac{1}{T}\sum_{k=1}^{T}\kappa^{-(k+S-L)}\right| \leq \frac{1}{T}\sum_{k=1}^{T}\left|\kappa^{-(k+S-L)}\right| \leq \frac{1}{T}\sum_{k=1}^{T}1 = 1 < \phi.$$

Hence, in this case, there cannot be any roots weakly outside the unit circle. Thus, by the fundamental theorem of algebra, the equation has $\max\{0, -(1+S-L)\}$ roots all strictly inside the unit circle.

Case 3: $1 + S - L < 0$ and $T + S - L > 0$

Multiplying the original equation by $\lambda^{-(1+S-L)}$ gives:

$$\frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1} = \phi\lambda^{-(1+S-L)}.$$

This equation has precisely the same roots of the original, since the original contained a term in λ^{1+S-L} and $1 + S - L < 0$. Now if $|\lambda| = 1$, then by the triangle inequality:

$$\left|\frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1}\right| \leq \frac{1}{T}\sum_{k=1}^{T}\left|\lambda^{k-1}\right| = \frac{1}{T}\sum_{k=1}^{T}1 = 1.$$

Also, if $|\lambda| = 1$, then:

$$\left|-\phi\lambda^{-(1+S-L)}\right| = \phi > 1.$$

Thus, for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$:

$$\left|\frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1}\right| < \left|-\phi\lambda^{-(1+S-L)}\right|.$$

This means that the original equation cannot have any roots with $|\lambda| = 1$. Additionally, by Rouché's theorem, this implies that the polynomial $\lambda \mapsto \frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1} - \phi\lambda^{-(1+S-L)}$ has the same number of zeros strictly inside the unit circle as the polynomial $\lambda \mapsto -\phi\lambda^{-(1+S-L)}$ (counting multiplicities). The latter polynomial has $-(1+S-L)$ roots inside the unit circle (all equal to zero). Therefore, both $\frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1} = \phi\lambda^{-(1+S-L)}$ and our original equation have $-(1+S-L) = \max\{0, -(1+S-L)\}$ roots strictly inside the unit circle.

Finally, note that $\frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1} = \phi\lambda^{-(1+S-L)}$ is a polynomial of degree $\max\{T-1, -(1+S-L)\}$, hence it has $\max\{T-1, -(1+S-L)\}$ roots in total, by the fundamental theorem of algebra. Hence our original equation has:

$$\begin{aligned}\max\{T-1, -(1+S-L)\} - [-(1+S-L)] &= T - 1 + (1+S-L) = T + S - L \\ &= \max\{0, T+S-L\}\end{aligned}$$

roots strictly outside the unit circle. This completes the proof.

E.9 Uniqueness and positivity of the multiperiod bond solution

We are interested in the solution of the difference equation:

$$A_j = \frac{1}{\phi}\mathbb{1}[j=0] + \frac{1}{\phi T}\sum_{k=1}^{T}A_{j-k-S+L}.$$

To understand this difference equation, first let $\ell^\infty(\mathbb{Z})$ be the space of bounded sequences with indices in $\mathbb{Z}$. This is a complete normed space under the sup-norm. Then define an operator $\mathcal{T}: \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by:

$$\left(\mathcal{T}(\tilde{A})\right)_j = \frac{1}{\phi}\mathbb{1}[j=0] + \frac{1}{\phi T}\sum_{k=1}^{T}\tilde{A}_{j-k-S+L},$$

for all $\tilde{A} \in \ell^\infty(\mathbb{Z})$ and $j \in \mathbb{Z}$.

Note that for $A^{(1)}, A^{(2)} \in \ell^\infty(\mathbb{Z})$ and $j \in \mathbb{Z}$:

$$\left(\mathcal{T}(A^{(1)}) - \mathcal{T}(A^{(2)})\right)_j = \frac{1}{\phi T} \sum_{k=1}^{T} \left(A^{(1)}_{j-k-S+L} - A^{(2)}_{j-k-S+L}\right),$$

so:

$$\left|\left(\mathcal{T}(A^{(1)}) - \mathcal{T}(A^{(2)})\right)_j\right| \leq \frac{1}{\phi T} \sum_{k=1}^{T} \left|A^{(1)}_{j-k-S+L} - A^{(2)}_{j-k-S+L}\right| \leq \frac{1}{\phi T} \sum_{k=1}^{T} \left\|A^{(1)} - A^{(2)}\right\|_\infty$$
$$= \frac{1}{\phi} \left\|A^{(1)} - A^{(2)}\right\|_\infty.$$

This means that for all $A^{(1)}, A^{(2)} \in \ell^\infty(\mathbb{Z})$:

$$\left\|\mathcal{T}(A^{(1)}) - \mathcal{T}(A^{(2)})\right\|_\infty \leq \frac{1}{\phi} \left\|A^{(1)} - A^{(2)}\right\|_\infty,$$

and hence that $\mathcal{T}$ is a contraction mapping, as $\phi > 1$. Thus, by the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point, which must be our desired $A = (A_j)_{j \in \mathbb{Z}}$. Furthermore, the Banach fixed point theorem implies that if we define $A^{(0)}_j := 0$ for all $j \in \mathbb{Z}$, and $A^{(n+1)} := \mathcal{T}(A^{(n)})$ for all $n \in \mathbb{N}$, then $A^{(n)} \to A$ (under the sup norm) as $n \to \infty$.

Now, suppose $\tilde{A} \in \ell^\infty(\mathbb{Z})$ with $\tilde{A}_j \geq 0$ for all $j \in \mathbb{Z}$. Then, by the definition of $\mathcal{T}$, $\left(\mathcal{T}(\tilde{A})\right)_j \geq 0$ for all $j \in \mathbb{Z}$. Hence, as $A^{(0)}_j \geq 0$ for all $j \in \mathbb{Z}$, by induction, $A^{(n)}_j \geq 0$ for all $j \in \mathbb{Z}$. Therefore, as $A^{(n)} \to A$ as $n \to \infty$, $A_j \geq 0$ for all $j \in \mathbb{Z}$.

E.10 Approximate uniqueness with endogenous wedges and multi-period bonds

Under the setup of Section 3, if we define $B := \sum_{j=-\infty}^{\infty} A_j$, then:

$$B = \frac{1}{\phi} + \sum_{j=-\infty}^{\infty} \frac{1}{\phi T} \sum_{k=1}^{T} A_{j-k-S+L} = \frac{1}{\phi} + \frac{1}{\phi T} \sum_{k=1}^{T} \sum_{j=-\infty}^{\infty} A_{j-k-S+L} = \frac{1}{\phi} + \frac{1}{\phi T} \sum_{k=1}^{T} B = \frac{1}{\phi}(1 + B).$$

Thus, $B = \frac{1}{\phi - 1}$. This will be sufficient to establish that $\pi_t \approx \pi^*_t$ for large ϕ, even when $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is endogenous, by an identical to argument to that of Subsection 2.2.

In particular, suppose we assume that $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is stationary, and that there exists some $\bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2, \bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2 \geq 0$ such that for any stationary solution for $\pi_t - \pi^*_t$,

$\left|\mathbb{E}(\nu_{t+S|t} - \bar{\nu}_{t+S|t})\right| \le \bar{\mu}_0 + \bar{\mu}_1 |\mathbb{E}(\pi_t - \pi_t^*)| + \bar{\mu}_2 \operatorname{Var}(\pi_t - \pi_t^*)$ and $\operatorname{Var}(\nu_{t+S|t} - \bar{\nu}_{t+S|t}) \le \bar{\gamma}_0 + \bar{\gamma}_1 |\mathbb{E}(\pi_t - \pi_t^*)| + \bar{\gamma}_2 \operatorname{Var}(\pi_t - \pi_t^*)$, for all $t \in \mathbb{Z}$ and $j, k \in \mathbb{N}$. This assumption is very mild, as already discussed in Subsection 2.2. Then, following the argument of that subsection (and using the fact that $A_j \ge 0$ for all $j \in \mathbb{Z}$):

$$|\mathbb{E}(\pi_t - \pi_t^*)| \le \frac{\bar{\mu}_0 + \bar{\mu}_1 |\mathbb{E}(\pi_t - \pi_t^*)| + \bar{\mu}_2 \operatorname{Var}(\pi_t - \pi_t^*)}{\phi - 1},$$

and:

$$\operatorname{Var}(\pi_t - \pi_t^*) \le \frac{\bar{\gamma}_0 + \bar{\gamma}_1 |\mathbb{E}(\pi_t - \pi_t^*)| + \bar{\gamma}_2 \operatorname{Var}(\pi_t - \pi_t^*)}{(\phi - 1)^2}.$$

So, for sufficiently large ϕ:

$$|\mathbb{E}(\pi_t - \pi_t^*)| \le \frac{[(\phi-1)^2 - \bar{\gamma}_2]\bar{\mu}_0 + \bar{\mu}_2 \bar{\gamma}_0}{(\phi - 1 - \bar{\mu}_1)[(\phi-1)^2 - \bar{\gamma}_2] - \bar{\mu}_2 \bar{\gamma}_1} = O\left(\frac{1}{\phi}\right),$$
$$\operatorname{Var}(\pi_t - \pi_t^*) \le \frac{(\phi - 1 - \bar{\mu}_1)\bar{\gamma}_0 + \bar{\mu}_0 \bar{\gamma}_1}{(\phi - 1 - \bar{\mu}_1)[(\phi-1)^2 - \bar{\gamma}_2] - \bar{\mu}_2 \bar{\gamma}_1} = O\left(\frac{1}{\phi^2}\right).$$

Hence, as $\phi \to \infty$, $\mathbb{E}(\pi_t - \pi_t^*) \to 0$ and $\operatorname{Var}(\pi_t - \pi_t^*) \to 0$.

E.11 Uniqueness of the solution for the modified inflation target

We are examining the modified inflation target monetary rule from Subsection 4.2. As in Holden (2021), we work under perfect foresight, with the initial state r_0 given. We assume the rest of the model is given by equations (4) and (5), so we have the following equations for $t > 0$:

$$\pi_t = \beta \pi_{t+1} + \kappa x_t,$$
$$x_t = \delta x_{t+1} - \varsigma (r_t - n_t),$$
$$i_t = r_t + \breve{\pi}_{t+1}^* + \phi(\pi_t - \breve{\pi}_t^*),$$
$$i_t = r_t + \pi_{t+1},$$
$$\breve{\pi}_t^* = \max\{-r_{t-1} + \epsilon, \pi_t^*\}.$$

We are interested in the constraint in the definition of $\breve{\pi}_t^*$, which we note can be rewritten as the pair of equations:

$$z_t = \breve{\pi}_t^* + r_{t-1} - \epsilon, \qquad z_t = \max\{0, \pi_t^* + r_{t-1} - \epsilon\},$$

where z_t is an auxiliary variable. We assume that $\pi_t^* + r_{t-1} - \epsilon > 0$ in steady-state, so that in steady-state, $\breve{\pi}_t^* = \pi_t^*$.

The results of Holden (2021) imply that conditional on z_t eventually converging to its positive steady state value, we can prove uniqueness via replacing the second equation for z_t just given with:

$$z_t = \pi_t^* + r_{t-1} - \epsilon + y_t,$$

where y_t is an exogenous forcing process. This implies that in equilibrium:

$$\pi_t = \check{\pi}_t^* = \pi_t^* + y_t,$$
$$x_t = \frac{1}{\kappa}[(\pi_t^* + y_t) - \beta(\pi_{t+1}^* + y_{t+1})],$$
$$r_t = n_t + \frac{1}{\kappa\varsigma}[-(\pi_t^* + y_t) + (\beta+\delta)(\pi_{t+1}^* + y_{t+1}) - \beta\delta(\pi_{t+2}^* + y_{t+2})],$$
$$z_t = \pi_t^* - \epsilon + y_t + n_{t-1} + \frac{1}{\kappa\varsigma}[-(\pi_{t-1}^* + y_{t-1}) + (\beta+\delta)(\pi_t^* + y_t) - \beta\delta(\pi_{t+1}^* + y_{t+1})],$$

from, respectively, the equations for z_t, the monetary rule and Fisher equation, the Phillips curve, the Euler equation, and the first equation for z_t.

Holden (2021) shows that uniqueness is determined by the determinants of the principal sub-matrices of the so-called "M" matrix for the model, which, here, contains the partial derivatives of z_t (t in rows) with respect to y_s (s in columns). We assume that π_t^* is exogenous. Thus, the M matrix is Toeplitz and tridiagonal, with $-\frac{1}{\kappa\varsigma}, 1 + \frac{\beta+\delta}{\kappa\varsigma}, -\frac{\beta\delta}{\kappa\varsigma}$ on the left, main and right diagonals respectively. Hence, by standard results, the eigenvalues of any $S \times S$ principal sub-matrix of M are given by $1 + \frac{\beta+\delta}{\kappa\varsigma} + \frac{2}{\kappa\varsigma}\sqrt{\beta\delta}\cos\left(\frac{s\pi^\circ}{S+1}\right)$ for $s \in \{1, \dots, S\}$, where π° is the mathematical constant usually denoted π. These are real if and only if $\beta\delta \geq 0$. They are positive for all S at least if $\kappa\varsigma > 0$. In this case, the minimum eigenvalue is greater than 1, so the determinant is also greater than 1, hence this is not knife-edge positivity: small changes to the model will not change this result. Thus, with π_t^* exogenous, $\pi_t = \check{\pi}_t^*$ is robustly the unique solution conditional on the terminal condition.

E.12 Optimal consumption with perpetuities and a permanent ZLB

For the sake of illustration, we adopt the simple parametric set-up used in Appendix B.1. It is clear our results are not specific to this set-up, however.

We suppose the representative household supplies one unit of labour, inelastically. Production of the final good is given by:

$$y_t = l_t (= 1).$$

In period t, the representative household maximises:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \log c_{t+k},$$

subject to the budget constraint:

$$P_t c_t + A_t + Q_t B_t + P_t \tau_t = P_t y_t + I_{t-1} A_{t-1} + B_{t-1}(1 + \omega Q_t),$$

where c_t is consumption, τ_t are real lump sum taxes, P_t is the price of the final good, A_t is the number of one period nominal bonds purchased by the household at t, which each return I_t in period $t+1$, Q_t is the price of a long (geometric coupon) bond and B_t are the number of units of this long bond purchased by the household at t. One unit of the period t long bond bought at t returns \$1 at $t+1$, along with $\omega \in (0,1]$ units of the period $t+1$ bond.

The household first order conditions imply:

$$1 = \beta I_t \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}},$$

$$Q_t = \beta \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}} (1 + \omega Q_{t+1}).$$

The household transversality conditions are that:

$$\lim_{k\to\infty} \beta^k \mathbb{E}_t \frac{A_{t+k}}{P_{t+k} c_{t+k}} = 0,$$

$$\lim_{k\to\infty} \beta^k \mathbb{E}_t \frac{Q_{t+k} B_{t+k}}{P_{t+k} c_{t+k}} = 0,$$

but we do not assume the second is necessary when $\omega = 1$. (The necessity of the transversality constraint when $\omega < 1$ follows from the following test given in Kamihigashi (2006), and formally proven in Kamihigashi (2003): "Shift the entire optimal path [for the state variable] downward by a small fixed proportion. Does it reduce the value of the objective function by only a finite amount? If so, the transversality condition is necessary.")

The government issues no one period bonds, so:

$$A_t = 0.$$

The government fixes the supply of long-bonds at:

$$B_t = B_t^* := B_{-1}\omega^{t+1}.$$

The central bank pegs nominal interest rates at the ZLB, meaning:

$$I_t = 1.$$

The final goods market clears, so:

$$y_t = c_t = 1.$$

Thus, from the household budget constraint, we have the following government budget constraint:

$$Q_t B_t^* + P_t \tau_t = B_{t-1}^*(1 + \omega Q_t).$$

We assume that the government adjusts taxes τ_t period by period to ensure this always holds (i.e., fiscal policy is passive and Ricardian). Thus:

$$P_t \tau_t = B_{-1}\omega^t.$$

Let $\Pi_t := \frac{P_t}{P_{t-1}}$, then from market clearing and the Euler equation for nominal bonds:

$$1 = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1}}.$$

So, from the Euler equation for the long bond:

$$Q_t = \frac{1}{1-\omega} + \lim_{k\to\infty} \omega^k \beta^k \mathbb{E}_t \left[\prod_{j=1}^{k} \frac{1}{\Pi_{t+j}} \right] Q_{t+k} \geq \frac{1}{1-\omega},$$

with equality when $\omega < 1$ as the transversality constraint definitely holds in that case. But, when $\omega = 1$, this says $Q_t \geq \infty$, so $Q_t = \infty$, hence $Q_t = \frac{1}{1-\omega}$ for all $\omega \in [0,1]$.

Now let:

$$b_t := \frac{Q_t B_t}{P_t},$$

then from the budget constraint:

$$P_t c_t + P_t b_t + P_t \tau_t = P_t + \frac{P_{t-1} b_{t-1}}{Q_{t-1}}(1 + \omega Q_t) = P_t + P_{t-1} b_{t-1},$$

and thus:

$$c_t + b_t + \tau_t = 1 + \frac{b_{t-1}}{\Pi_t}.$$

It is instructive to re-solve the original household problem under this rewritten budget constraint. This must have the same solution as the original problem. In particular, consider the problem of maximising:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \log c_{t+k},$$

subject to:

$$c_t + b_t + \tau_t = 1 + \frac{b_{t-1}}{\Pi_t},$$

by choosing $c_t, c_{t+1}, \dots, b_t, b_{t+1}, \dots$. This is the "textbook" cake eating problem with exogenous income, $1 - \tau_t$, and gross interest rate $\frac{1}{\Pi_t}$. The Euler equation is:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1} c_{t+1}},$$

and the (always necessary) transversality constraint states that:

$$\lim_{k \to \infty} \beta^k \mathbb{E}_t \frac{b_{t+k}}{c_{t+k}} = 0.$$

Additionally, the government budget constraint can be rewritten as:

$$\tau_t = (1 - \omega) b_{-1} \left[\prod_{s=0}^{t} \frac{1}{\Pi_s} \right] \omega^t.$$

We know that in equilibrium, market clearing implies $c_t = 1$, but for now, we will "forget" this fact, and merely suppose that $c_t = c$ for all t, for some $c > 0$. This satisfies the Euler equation as:

$$\frac{1}{c} = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1} c} = \frac{1}{c},$$

as $1 = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1}}$. Then transversality simplifies to:

$$\lim_{k \to \infty} \beta^k \mathbb{E}_t b_{t+k} = 0,$$

and the budget constraint gives:

$$\begin{aligned} b_t &= \sum_{k=0}^{t} \left[\prod_{j=0}^{k-1} \frac{1}{\Pi_{t-j}} \right] (1 - c_{t-k} - \tau_{t-k}) + \left[\prod_{j=0}^{t} \frac{1}{\Pi_{t-j}} \right] b_{-1} \\ &= (1 - c) \sum_{k=0}^{t} \prod_{s=t-k+1}^{t} \frac{1}{\Pi_s} + \omega^{t+1} b_{-1} \prod_{s=0}^{t} \frac{1}{\Pi_s}, \end{aligned}$$

by the simplified government budget constraint previously derived. Hence, since $1 = \mathbb{E}_t \beta \frac{1}{\Pi_{t+1}}$:

$$\beta^t \mathbb{E}_0 b_t = (1-c)\frac{1-\beta^{t+1}}{1-\beta} + \omega^{t+1} b_{-1} \frac{1}{\Pi_0},$$

so, by the period 0 transversality constraint:

$$0 = \lim_{t\to\infty} \beta^t \mathbb{E}_0 b_t = \frac{1-c}{1-\beta} + b_{-1}\frac{1}{\Pi_0}\lim_{t\to\infty} \omega^{t+1}.$$

If $\omega \in (0,1)$, then this implies that $c = 1$ as expected. However, if $\omega = 1$, then:

$$c = 1 + (1-\beta)\frac{b_{-1}}{\Pi_0}.$$

Thus, if Π_0 is finite, then $c > 1$, violating the market clearing condition. The only way to restore market clearing is if Π_0 is infinite. This is intuitive, as when $\omega = 1$, households have infinite nominal wealth, which cannot fail to push up prices.

E.13 Solution properties of first welfare example

Recall, that for $k > 1$ the solution must satisfy the recurrence relation:

$$\theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \beta\frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

The characteristic equation of this recurrence relationship has roots:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right) \pm \sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2 - \left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$= \frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right) \pm \sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}.$$

The positive root satisfies:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)+\sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$> \frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)+\sqrt{\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$= \frac{1+\frac{\lambda}{\kappa^2}-\beta(1-\beta)\frac{\lambda}{\kappa^2}}{\beta\frac{\lambda}{\kappa^2}} > \frac{1+\frac{\lambda}{\kappa^2}-(1-\beta)\frac{\lambda}{\kappa^2}}{\beta\frac{\lambda}{\kappa^2}} = 1+\frac{1}{\beta\frac{\lambda}{\kappa^2}} > 1.$$

The negative root satisfies:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2-\left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$> \frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} = 0,$$

and:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$< \frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}} = 1.$$

Hence, the positive root is greater than 1, while the negative root is in $(0,1)$. Thus for $k \geq 1$:

$$\theta_k = \theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2-\left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right]^{k-1}.$$

Hence, θ_0, θ_1 and θ_2 are the unique solution of the three linear (in θ_0, θ_1 and θ_2) equations:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_2 = \theta_1\left[\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2\frac{\lambda}{\kappa^2}\right)^2 - \left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right].$$

E.14 Solution under discretion of first welfare example

Under discretion, we have the standard first order condition:

$$\pi_t + \frac{\lambda}{\kappa}x_t = 0,$$

i.e.:

$$\kappa\sum_{k=0}^{\infty}\theta_k\omega_{t-k} + \frac{\lambda}{\kappa}\sum_{k=0}^{\infty}(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} = 0,$$

so:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\forall k \geq 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) = 0.$$

The latter recurrence relation has the general solution $\theta_k = \theta_1\left(\frac{\kappa^2}{\beta\lambda} + \frac{1}{\beta}\right)^{k-1}$, which is explosive as $\beta < 1$. Thus, we must have $\theta_1 = \theta_2 = \cdots = 0$. Hence, $\theta_0 = \frac{\lambda}{\lambda+\kappa^2}$.

E.15 Solution under the timeless perspective of first welfare example

The timeless perspective (Woodford 1999) leads to the first order condition:

$$\pi_t + \frac{\lambda}{\kappa}(x_t - x_{t-1}) = 0,$$

i.e.:

$$\kappa\sum_{k=0}^{\infty}\theta_k\omega_{t-k} + \frac{\lambda}{\kappa}\left[\sum_{k=0}^{\infty}(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} - \sum_{k=1}^{\infty}(\theta_{k-1} - \beta\theta_k - \mathbb{1}[k-1=0])\omega_{t-k}\right] = 0,$$

so:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\forall k > 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

The roots of the characteristic equation corresponding to the latter recurrence relation are:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) \pm \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}.$$

The positive root satisfies:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) + \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} > \frac{\frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}}{2\beta\frac{\lambda}{\kappa^2}} = \frac{1+\beta}{2\beta} > 1.$$

The negative root satisfies:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$> \frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} = 0,$$

and:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$=\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{1+(1-\beta)^2\left(\frac{\lambda}{\kappa^2}\right)^2+2(1+\beta)\frac{\lambda}{\kappa^2}}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$<\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{1+(1-\beta)^2\left(\frac{\lambda}{\kappa^2}\right)^2+2(1-\beta)\frac{\lambda}{\kappa^2}}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$=\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1-\beta)\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}=\frac{2\beta\frac{\lambda}{\kappa^2}}{2\beta\frac{\lambda}{\kappa^2}}=1.$$

Hence, the positive root is greater than 1, while the negative root is in $(0,1)$. Thus for $k \geq 1$:

$$\theta_k=\theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right]^{k-1}.$$

Hence, θ_0, θ_1 and θ_2 are the unique solution of the three linear (in θ_0, θ_1 and θ_2) equations:

$$\theta_0+\frac{\lambda}{\kappa^2}(\theta_0-\beta\theta_1-1)=0,$$
$$\theta_1+\frac{\lambda}{\kappa^2}(\theta_1-\beta\theta_2)-\frac{\lambda}{\kappa^2}(\theta_0-\beta\theta_1-1)=0,$$
$$\theta_2=\theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right].$$

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